# ISOTOPY GROUPS

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ABSTRACT. For any mapping $f: V \to M$ (not necessarily an embedding), where $V$ and $M$ are differentiable manifolds without boundary of dimensions $k$ and $n$, respectively, $V$ compact, let $[V \subset M]_f = \pi_1(M^V, E, f)$, i.e., the set of isotopy classes of embeddings with a specific homotopy to $f$ ($E$ = space of embeddings). The purpose of this paper is to enumerate $[V \subset M]_f$. For example, if $k > 3$, $n = 2k$, and $M$ is simply connected, $[S^k \subset M]_f$ corresponds to $\pi_2 M$ or $\pi_2 M \otimes Z_2$, depending on whether $k$ is odd or even. In the metastable range, i.e., $3(k + 1) > 2n$, a natural Abelian affine structure on $[V \subset M]_f$ is defined: if, further, $f$ is an embedding $[V \subset M]_f$ is then an Abelian group. The set of isotopy classes of embeddings homotopic to $f$ is the set of orbits of the obvious left action of $\pi_1(M^V, f)$ on $[V \subset M]_f$.

A spectral sequence is constructed converging to a theory $H^*(f)$. If $3(k + 1) < 2n$, $H^0(f) \cong [V \subset M]_f$ provided the latter is nonempty. A single obstruction $\Gamma(f) \in H^1(f)$ is also defined, which must be zero if $f$ is homotopic to an embedding; this condition is also sufficient if $3(k + 1) < 2n$. The $E_2$ terms are cohomology groups of the reduced deleted product of $V$ with coefficients in sheaves which are not even locally trivial. $[S^k \subset M]_f$ is specifically computed in terms of generators and relations if $n = 2k$, $k > 3$ (Theorem 6.0.2).

1. **Introduction.** In this paper, in some respects a sequel to [7], we attack the general problem of classifying, up to isotopy, embeddings of a compact $k$-manifold $V$ in an $n$-manifold $M$ in the metastable range, i.e., where $3(k + 1) < 2n$.

Differentiable shall mean infinitely differentiable, manifold shall mean differentiable manifold without boundary (either compact or open), with a countable base; embedding shall mean differentiable embedding, and isotopy shall mean homotopy of embeddings.

Let $[V \subset M]$ denote the set of isotopy classes of embeddings of $V$ in $M$. Computation of $[V \subset M]$ is the ultimate goal. Unfortunately, $[V \subset M]$ has no convenient algebraic structure. Thus we introduce a new object, $[V \subset M]_f$, where $f: V \to M$ is a specific map. A homotopy $f_t: V \to M, 0 < t < 1$, is

called an embedding-homotopy (or $e$-homotopy) of $f$ if $f_0 = f$ and $f_1$ is an embedding. If $f_{u,t}: V \to M$, for $0 \leqslant t$, $u \leqslant 1$ is a two-parameter homotopy such that, for all $u$, $f_{u,0} = f$ and $f_{u,1}$ is an embedding, we say that the $e$-homotopies $\{f_{0,t}\}$ and $\{f_{1,t}\}$ are isotopic. $[V \subset M]_f$ is then defined to be the set of isotopy classes of $e$-homotopies of $f$. If $\{f_t\}$ is an $e$-homotopy of $f$, let $[f_t] \in [V \subset M]_f$ be its isotopy class. It shall be shown that $[V \subset M]_f$ is an Abelian affine group in the metastable range, and an actual Abelian group if $f$ is an embedding.

If $2k - n \leqslant 0$, the group $[S^k \subset M^n]_f$ is, in this paper, expressed in terms of generators and relations, involving only $\pi_1 M$ if $n = 2k + 1$, and involving both $\pi_1 M$ and $\pi_2 M$ if $n = 2k$. For $n > 2k + 1$, $[V^k \subset M^n]_f = 0$.

Generally (although not shown in this paper) the affine structure of $[V^k \subset M^n]_f$ in the metastable range depends only on the homotopy of $M$ through dimension $2k - n + 2$, as well as on $V$.

The case $n = 2k + 1$ was done in [7], but the result was incorrectly stated. See Theorem 6.0.1 for the correct version. (Another error in [7], an invalid proof of Theorems 3.3.1 and 3.3.2, is corrected here in §8. The error was pointed out by the referee of this paper.) The case $n = 2k$ is computed for the first time in this paper, and involves evaluation of one nonzero differential and one nontrivial extension in a spectral sequence whose $E_2$ terms are cohomology groups of the reduced deleted product of $S^k$ with coefficients in sheaves which do not, in general, have a local product structure. See Theorem 6.0.2 for the general result. Some specific cases are as follows (where $f$ is any embedding):

THEOREM 1.0.1. *If $M = M^{2k}$ is simply connected, $k \geqslant 3$, then:*

$$[S^k \subset M]_f \cong \begin{cases} \pi_2 M \otimes Z_2 & \text{if } k \text{ is even,} \\ \pi_2 M & \text{if } k \text{ is odd.} \end{cases}$$

THEOREM 1.0.2. *If $k \geqslant 3$:*

$$[S^k \subset P^2 \times R^{2k-2}]_f \cong \begin{cases} Z_2 + Z_2 & \text{if } k \text{ is even,} \\ Z + Z + Z_2 & \text{if } k \text{ is odd.} \end{cases}$$

THEOREM 1.0.3. *If $k \geqslant 3$:*

$$[S^k \subset P^{2k}]_f \cong \begin{cases} 0 & \text{if } k \equiv 0 \bmod 4, \\ Z_4 & \text{if } k \equiv 1 \bmod 4, \\ Z_2 & \text{if } k \equiv 2 \bmod 4, \\ Z_2 + Z_2 & \text{if } k \equiv 3 \bmod 4. \end{cases}$$

Let $\pi = \pi_1(M^V, f)$. We may define a left action:

$$\mu: \pi \times [V \subset M]_f \to [V \subset M]_f$$

as follows: if $\{f_t\}$ is an $e$-homotopy of $f$ and if $\{f_t'\}$ is a self-homotopy of $f$ representing $a \in \pi$, then $\mu(a, [f_t]) = [f_t'']$ where $f_t'' = f_{2t}$ or $f_{2t-1}'$, depending on the value of $t$. The action $\mu$ respects the affine structure of $[V \subset M]_f$, i.e., for each $a \in \pi$, $\mu(a, )$ is an affine automorphism (cf. Theorem 2.7.1).

Now let $\phi: [V \subset M]_f \to [V \subset M]$ be the forgetful function, i.e., if $\{f_t\}$ is an $e$-homotopy of $f$, $\phi[f_t] = [f_1]$, the isotopy class containing $f_1$. Trivially, the reader can convince himself of the following:

(i) Every element of $[V \subset M]$ lies in the image of $\phi$ for some choice of $f$.

(ii) If $x, y \in [V \subset M]_f$, then $\phi x = \phi y$ if and only if $\mu(a, x) = y$ for some $a \in \pi$.

Classification, up to isotopy, of embeddings of $V$ in $M$ in the stable range may thus be reduced to the following two problems:

I. Compute the affine structure of $[V \subset M]_f$, for each $f: V \to M$.

II. Determine the action $\mu$ of $\pi_1(M^V, f)$ on $[V \subset M]_f$.

It is to the first of these problems that this paper is addressed.

In §2, we use Haefliger's results [4] to define the affine structure on $[V \subset M]_f$. An alternative, equivalent definition of the affine structure is given in §4; this definition is much more geometric and easier to comprehend, but it is the Haefliger definition that yields to computation, by homotopy methods. In §3, a spectral sequence is defined, in §5, the sheaves we need are computed, and in §6 the spectral sequence is fully worked in the special cases.

Using somewhat different methods, Dax [2] and Salomonsen [12] have obtained results on the same and similar problems.

## 2. The affine structure of $[V \subset M]_f$.

2.1. *Pair fibrations and weak pair fibrations*. Following [11], we say that a map $p: (E, E_0) \to (X, X_0)$ is a *pair fibration* if for any given map $h: (Y, Y_0) \to (E, E_0)$ and every homotopy $f_t: (Y, Y_0) \to (X, X_0)$, $0 < t < 1$, such that $f_0 = ph$, there exists a homotopy $h_t: (Y, Y_0) \to (E, E_0)$ such that $h_0 = h$ and $ph_t = f_t$ for all $t$.

A *section* of a pair fibration is a map $g: (X, X_0) \to (E, E_0)$ such that $pg = 1$ (the identity). A *lifting* of $f: (Y, Y_0) \to (X, X_0)$ to $p$ is a map $h: (Y, Y_0) \to (E, E_0)$ such that $ph = f$. Sections and liftings are said to be *homotopic* if they are homotopic *as* sections or liftings (i.e., fiber homotopic).

All the theorems in §§2 and 3 which deal with sections of pair fibrations are equivalent to corresponding theorems which deal with liftings, since a lifting of $f$ to $p$ corresponds to a section of $f^{-1}p$ (where $f^{-1}$ is the usual pullback construction).

If $(X, X_0)$ is a topological pair, we say that $(A, A_0)$ is a *sub-pair* of $(X, X_0)$ if $A \subset X$ and $A_0 = A \cap X_0$. Then $((X, X_0), (A, A_0))$ is a *pair of pairs*. If $X$ is a C.W. complex, and $X_0$ and $A$ are subcomplexes, it is a C.W. pair of pairs.

For the following definitions, let (as needed) $f: ((Y, Y_0), (B, B_0)) \to$

$((X, X_0), (A, A_0))$ be a map of pairs of pairs, let $z\colon (A, A_0) \to (E, E_0)$ be a partial section of $p$, let $w\colon (B, B_0) \to (E, E_0)$ be a partial lifting of $f$ to $p$.

DEFINITION 2.1.1. Let $[(X, X_0); p]$ be the set of homotopy classes of sections of $p$.

DEFINITION 2.1.2. Let $[(Y, Y_0); p]_f$ be the set of homotopy classes of liftings of $f$ to $p$.

DEFINITION 2.1.3. Let $[(X, X_0), (A, A_0); p]^z$ be the set of rel$(A, A_0)$ homotopy classes of sections of $p$ which extend $z$. If $p$ has a standard section, say $s\colon (X, X_0) \to (E, E_0)$, then we shall always presume (unless otherwise stated) that $z = s|(A, A_0)$ and suppress $z$ in the notation.

DEFINITION 2.1.4. Let $[(Y, Y_0), (B, B_0); p]_f^w$ be the set of rel$(B, B_0)$ homotopy classes of liftings of $f$ to $p$ which extend $w$. If $p$ has a standard section, $s$, presume that $w = sf|(B, B_0)$ and suppress $w$ in the notation.

The following lemma will be important later.

LEMMA 2.1.5. *If $((X, X_0), (A, A_0))$ is a C.W. pair of pairs, $p$ is a pair fibration over $(X, X_0)$, $\{z_t\}$ is a homotopy of partial sections of $p$ over $(A, A_0)$, there is a one-to-one correspondence $[(X, X_0), (A, A_0); p]^{z_0} \cong [(X, X_0), (A, A_0); p]^{z_1}$ such that, whenever $\{g_t\}$ is a homotopy of sections of $p$ which extends $\{z_t\}$, $[g_0]$ (the homotopy class containing $g_0$) corresponds to $[g_1]$.*

PROOF. By induction on the skeleta of $X$. We omit the details.

2.2. *Fiberwise suspensions.* Choose standard spheres and balls $S^{N-1} \subset B^N$ for all $N \geq 0$ (where $S^{-1}$ is empty) such that $S^0 * S^{N-1} = S^N$ and $S^0 * B^N = B^{N+1}$ (where $* = $ join). Let each sphere have a South pole and a North pole, preserved under inclusion $S^N \subset S^{N+1}$. The South pole will be considered to be the base point of each sphere.

If $p\colon E \to X$ is a fibration, let $S^N p\colon S_X^N E \to X$, where $S_X^N E = S^N *_X E$, the fiberwise join of $p\colon E \to X$ with the trivial fibration $X \times S^N \to X$. We give $S^N *_X E$ the strong topology, thus insuring that $S^N p$ is a fibration [5]. Note that $S^0 p = p$, and that for $N \geq 1$, $S^N p$ has two standard sections, the South polar and the North polar, denoted $s_0$ and $s_1$. If only one section is needed, $s_0$ shall be used. Note also that $Sp = S^1 p$ is the fiberwise two point suspension of $p$.

Now, for any $N \geq 0$, let $P^N S^N p\colon P_X^N S_X^N \to X$ be defined as follows. For each $x \in X$, let $(P^N S^N p)^{-1}\{x\}$ be the set of all maps $\sigma\colon B^N \to (S^N p)^{-1}\{x\}$ such that $\sigma|S^{N-1}$ is simply the identity (recall that each fiber of $S^N p$ contains a copy of $S^{N-1}$). Let $P_X^N S_X^N E$ then have the topology of a subspace of $(S_X^N E)^{B^N}$ with the compact-open topology. We can define a fiber-preserving inclusion $P_X^N S_X^N E \subset P_X^{N+1} S_X^{N+1} E$ by identifying each $\sigma \in P_X^N S_X^N E$ with $S^0 * \sigma$.

Now if $M \geq 0$, $N \geq 1$, let $\Omega^M S^N p\colon \Omega_X^M S_X^N E \to X$ be given by

$(\Omega^M S^N p)^{-1}\{x\} = \Omega^M (S^N p)^{-1}\{x\}$, the $M$-fold loop space, which we take to be the set of maps $(B^M, S^{M-1}) \to ((S^N p)^{-1}\{x\}, s_0 x)$, for any $x \in X$. We use the compact open topology for $\Omega_X^M S_X^N E$, and again there is an inclusion $\Omega_X^M S_X^M E \subset \Omega_X^{M+1} S_X^{N+1} E$.

The above constructions generalize naturally to pair fibrations. If $p: (E, E_0) \to (X, X_0)$ is a pair fibration, we define:

$$S^N p: \left( S_X^N E, S_{X_0}^N E_0 \right) \to (X, X_0),$$

$$P^N S^N p: \left( P_X^N S_X^N E, P_{X_0}^N S_{X_0}^N E_0 \right) \to (X, X_0),$$

$$\Omega^M S^N p: \left( \Omega_X^M S_X^N E, \Omega_{X_0}^M S_{X_0}^N E_0 \right) \to (X, X_0).$$

2.3. *Groups and affine groups of sections and liftings.* Henceforth ·in this section, let $((X, X_0), (A, A_0))$ be a C.W. pair of pairs, and $p: (E, E_0) \to (X, X_0)$ a pair fibration.

THEOREM 2.3.1. *If $N \geqslant 1$, $M \geqslant 1$, then $[(X, X_0), (A, A_0); \Omega^M S^N p]$ is a group. Furthermore, it is Abelian if $M \geqslant 2$.*

We omit the proof of 2.3.1: simply follow, fiberwise, the usual proof that $[X, A; \Omega^M Y]$ is a group, Abelian if $M \geqslant 2$, if $Y$ is any pointed space.

*The affine structure.* Let $N \geqslant 1$, and fix a partial section of $P^N S^N p$ over $(A, A_0)$. We proceed to define a ternary operation, $\tau$, on $[(X, X_0), (A, A_0); P^N S^N p]^z$.

Let $U$ and $L$ be the upper and lower hemispheres, respectively, of $\partial B^N$. Let $B_1^N, B_2^N$, and $B_3^N$ be copies of $B^N$, with upper and lower hemispheres $U_1, L_1, U_2$, etc. Now choose a homeomorphism $\phi: B^N \to W = (B_1^N \cup B_2^N \cup B_3^N)/\sim$, where "$\sim$" identifies $U_1$ with $U_2$ and $L_2$ with $L_3$, and $\phi: L \cong L_1$, $\phi: U \cong U_3$. (Note that $\phi$ has degree 1 in homology onto $B_1^N$ and $B_3^N$, and $-1$ onto $B_2^N$.) Now every section of $P^N S^N p$ may be thought of as a map $X \times B^N \to S_X^N E$ satisfying the appropriate conditions. If $g_1, g_2$, and $g_3$ are sections of $P^N S^N p$, we define a new section of $P^N S^N p$ called $g_1 \wedge g_2^{-1} \wedge g_3$, by commutativity of the following diagram for each $i = 1, 2, 3$:

$$
\begin{array}{ccc}
X \times B^N & \xrightarrow{g_1 \wedge g_2^{-1} \wedge g_3} & S_X^N E \\
\downarrow{1_X \times \phi} & & \uparrow{g_i} \\
X \times W \quad \supset X \times B_i^N & \cong & X \times B^N
\end{array}
$$

Now if $g$ is any section, $g \wedge g^{-1} \wedge g$ is canonically homotopic to $g$ itself. (To see this, let $\chi: W \to B^N$ be the "triple folding" map obtained by identifying each $B_i^N$ with $B^N$. Now choose a rel $S^{N-1}$ homotopy of $\chi \circ \phi: B^N \to B^N$ with the identity; this gives a homotopy of $g$ with $g \wedge g^{-1} \wedge g = 1_X \times (\chi \circ \phi)$. This homotopy is functorial with respect to maps of pair fibrations.) Thus, since $z \wedge z^{-1} \wedge z$ is homotopic to $z$, we obtain, by Lemma

2.1.5, a one-to-one correspondence

$$\xi\colon \left[(X, X_0), (A, A_0); P^N S^N p\right]^{z \wedge z^{-1} \wedge z} \cong \left[(X, X_0), (A, A_0); P^N S^N p\right]^z.$$

Now define $\tau([g_1], [g_2], [g_3]) = [g_1][g_2]^{-1}[g_3] = \xi[g_1 \wedge g_2^{-1} \wedge g_3]$.

THEOREM 2.3.2. (I) *Under* $\tau$, $[(X, X_0), (A, A_0); P^N S^N p]^z$ *is an affine group. If* $N > 2$, *it is an Abelian affine group.* (II) *If* $[(X, X_0), (A, A_0); P^N S^N p]^z$ *is nonempty, its left action group is* $[(X, X_0), (A, A_0); \Omega^N S^N p]$.

PROOF. It is a routine exercise to show that $\tau$ is associative and satisfies cancellation, and thus is an affine structure. To obtain (II), consider the map $\eta\colon B^N \to Y = (B_1^N \cup B_2^N)/\sim \cong S^N$ where "$\sim$" identifies $S_1^{N-1}$ with $S_2^{N-1}$, and $\eta$ maps all of $S^{N-1}$ onto the common South pole, and is of degree 1 onto $B_1^N$, degree $-1$ onto $B_2^N$. Then let $[g_1][g_2]^{-1} = [g_1 \wedge g_2^{-1}]$, where, for $i = 1$ or 2, the following diagram commutes:

$$
\begin{array}{ccc}
X \times B^N & \xrightarrow{\;g_1 \wedge g_2^{-1}\;} & S_X^N E \\[4pt]
\downarrow{\scriptstyle 1_X \times \eta} & & \uparrow \\[4pt]
X \times Y \quad \supset X \times B_i^N & \cong & X \times B^N
\end{array}
$$

Clearly $g_1 \wedge g_2^{-1}$ is a section of $\Omega^N S^N p$, and verification of the remaining details is routine. (Hint: $z \wedge z^{-1}$ is canonically homotopic to a trivial section.)

The commutativity of $\tau$ in the case $N > 2$ now follows from 2.3.1, and we are done with the proof of 2.3.2.

In certain cases which turn out to be important later there is a geometric definition of the affine structure. Suppose that $U_1$ and $U_2$ are open subsets of $X$ such that $U_1 \cup U_2 = X$, and suppose that $z$ is a partial section of $P^N S^N p$ over $(A, A_0)$.

PROPOSITION 2.3.3. *If* $g_1$ *and* $g_2$ *are sections of* $\Omega^M S^N p$, *for* $M > 1$, $N > 1$, *such that* $g_1$ *and  are both trivial on* $(A, A_0)$, *and such that* $g_1|U_2$ *is trivial and* $g_2|U_1$ *is trivial, let* $g_3$ *be the section which agrees with* $g_1$ *on* $U_1$, *with* $g_2$ *on* $U_2$, *and is trivial elsewhere. Then* $[g_1][g_2] = [g_3]$.

PROOF. It is clear that $g_3$ is homotopic to $g_1 \wedge g_2$ on $X - U_1$ and also on $X - U_2$; they are both trivial on $U_1 \cap U_2$. Homotopies on the two closed sets may easily be chosen, and do not interfere.

PROPOSITION 2.3.4. *If* $g_1$, $g_2$, *and* $g_3$ *are sections of* $P^N S^N p$, *for* $N > 1$, *such that all agree with* $z$ *on* $(A, A_0)$, *and such that* $g_1$ *agrees with* $g_2$ *outside of* $U_2$, *and* $g_3$ *agrees with* $g_2$ *outside of* $U_3$, *let* $g_4$ *be the section where, for all* $x \in X$,

$$h_4(x) = \begin{cases} h_1(x) & \text{if } x \notin U_2, \\ h_2(x) & \text{if } x \in U_1 \cap U_2, \\ h_3(x) & \text{if } x \notin U_1. \end{cases}$$

*Then* $[h_1][h_2]^{-1}[h_3] = [h_4]$.

PROOF. Note that $h_1 \wedge h_2^{-1} \wedge h_3$ and $h_4$ are homotopic, albeit via different standard homotopies, on each of the two sets $U_1$ and $U_2$. By choosing a Urysohn function $X \to I$ which is 0 on the complement of $U_2$ and 1 on the complement of $U_1$, these homotopies can be made to blend smoothly into one another over $U_1 \cap U_2$, since on that region $h_1, h_2, h_3$, and $h_4$ all agree.

2.4. *Equivalence of classes of sections.* Let $((X, X_0), (A, A_0))$ be a C.W. pair of pairs, and let $p: (E, E_0) \to (X, X_0)$ and $p': (E', E_0') \to (X, X_0)$ be pair fibrations. Let $\gamma: (E, E_0) \to (E', E_0')$ be a fiber map, and $z: (A, A_0) \to (E, E_0)$ a partial section of $p$. Finally, define $z' = \gamma \cdot z$, a partial section of $p'$ over $(A, A_0)$. We have an obvious function, induced by composition with $\gamma$:

$$\gamma_\#: \left[(X, X_0), (A, A_0); p\right]^z \to \left[(X, X_0), (A, A_0); p'\right]^{z'}.$$

Under certain conditions on the homotopy of the fibers, $\gamma_\#$ is one-to-one or onto.

Let $E_x = p^{-1}\{x\}$, for any $x \in X$, and let $(E_0)_x = E_x \cap E_0$. Similarly, let $E_x' = (p')^{-1}\{x\}$ and $(E_0')_x = E_x' \cap E_0'$.

THEOREM 2.4.1. *Suppose that $X/A$ is finite dimensional. Let $n = \dim(X/(X_0 \cup A))$, and $n_0 = \dim(X_0/A_0)$. (I) Suppose that, for all $x \in (X - (X_0 \cup A))$, $\gamma_*: \pi_k(E_x) \to \pi_k(E_x')$ is one-to-one and onto for all $0 < k < n$ and onto for $k = n$; and that, for all $x \in (X_0 - A_0)$, $\gamma_*: \pi_k(E_0)_x \to \pi_k(E_0')_x$ is one-to-one and onto for all $0 < k < n_0$, and onto for $k = n_0$. Then $\gamma_\#$ is onto. (II) Suppose that, for all $x \in (X - (X_0 \cup A))$, $\gamma_*: \pi_k(E_x) \to \pi_k(E_x')$ is one-to-one and onto for all $0 < k \leq n$ and onto for $k = n + 1$; and that $\gamma_*: \pi_k(E_0)_x \to \pi_k(E_0')_x$ is one-to-one and onto for all $0 \leq k \leq n_0$ and onto for $k = n_0 + 1$ for all $x \in (X_0 - A_0)$. Then $\gamma_\#$ is one-to-one.*

PROOF. This theorem is simply a pair-fibration version of a well-known result in fibrations. (See, for example, Lemma 2.2 of [8].) The proof is unaltered by the pair nature, since it is done inductively one cell at a time.

DEFINITION 2.4.2. We say that $p$ is *n-connected* if, for all $k < n$, $\pi_k(E)_x = 0$ for all $x \in X$ and $\pi_k(E_0)_x = 0$ for all $x \in X_0$.

COROLLARY 2.4.3. *If $p$ is n-connected and $\dim(X/A) \leq 2n$, then $[(X, X_0), (A, A_0); p]^z$ is an Abelian affine group.*

PROOF. The inclusion $i: p \subset P^N S^N p$ induces isomorphism on the homotopy of the fibers up through dimension $2n$, and epimorphism in dimension $2n + 1$

(since each fiber of $P^N S^N p$ is of the same homotopy type as the $N$-fold loop space of the $N$-fold suspension of the corresponding fiber of $p$). Thus, by 2.4.1, $[(X, X_0), (A, A_0); p]^z \cong [(X, X_0), (A, A_0); P^N S^N p]^{iz}$ for all $N$. Apply 2.3.2, and we are done.

We remark that the Abelian affine structure on $[(X, X_0), (A, A_0); p]^z$, under the hypotheses of 2.4.3, can be alternatively defined in manner described in 2.3.4. Verification is trivial.

2.5. *The obstruction theory.* Let $((X, X_0), (A, A_0))$ be a C.W. pair of pairs, $p: (E, E_0) \to (X, X_0)$ a pair fibration, and $z$ a partial section of $p$ over $(A, A_0)$.

DEFINITION 2.5.1. For any integer $i$, let

$$H^i((X, X_0), (A, A_0); p) = \lim_{N \to \infty} \left[ (X, X_0), (A, A_0); \Omega^{N-i} S^N p \right],$$

the direct limit. Write $H^i((X, X_0); p)$ if $A$ is empty.

We remark in passing that $H^*( \ ; p)$ is a cohomology theory in a certain sense; and satisfies a version of the Eilenberg-Steenrod axioms. In particular, if $f: ((Y, Y_0), (B, B_0)) \to ((X, X_0), (A, A_0))$ is a map of C.W. pairs of pairs, there is an induced homomorphism

$$f^*: H^i((X, X_0), (A, A_0); p) \to H^i((Y, Y_0), (B, B_0); f^{-1}p),$$

and if $q: (F, F_0) \to (X, X_0)$ is another pair fibration and $\gamma: (E, E_0) \to (F, F_0)$ a fiber preserving map, there is an induced homomorphism

$$\gamma_\#: H^i((X, X_0), (A, A_0); p) \to H^i((X, X_0), (A, A_0); q).$$

REMARK 2.5.2. If $p$ is $n$-connected and $\dim(X/A) \leq 2n$, then $[(X, X_0), (A, A_0); p]^z$ (an Abelian affine group, by 2.4.3), if nonempty, has $H^0((X, X_0), (A, A_0); p)$ as its action group.

Thus, classification of sections of a pair fibration, under suitable dimensional restrictions, reduces to algebraic computation of a cohomology group. In the next section, we shall show how $H^*((X, X_0), (A, A_0); p)$ can be attacked by a familiar spectral sequence technique.

*The single obstruction.* We now consider the question of whether $z$ can be extended to a section of $p$. Recall that $Sp$ has two sections, $s_0$ and $s_1$, the South and North polar sections. Now $S_X E$ may be obtained from $E \times I$ by collapsing the ends in the appropriate manner (although with the strong, not the quotient topology). Let $u_t x = [zx, t] \in S_A E$ for all $x \in A$. By 2.1.5, $\{u_t\}$ gives a one-to-one correspondence:

$$\sigma: \left[ (X, X_0), (A, A_0); Sp \right]^{s_1 | A} \cong \left[ (X, X_0), (A, A_0); Sp \right].$$

DEFINITION 2.5.3. Let $\gamma(p) = \sigma[s_1] \in [(X, X_0), (A, A_0); Sp]$, the *primitive single obstruction* to section of $p$ extending $z$.

DEFINITION 2.5.4. Let $\Gamma(p) \in H^1((X, X_0), (A, A_0); p)$ be the image of $\gamma(p)$ in the direct limit, the *single obstruction* to section of $p$ extending $z$.

THEOREM 2.5.5. (I) *There exists a section of p extending* $z \Rightarrow \gamma(p) = 0 \Rightarrow$ $\Gamma(p) = 0$. (II) *Suppose that p is n-connected and* $\dim(X/A) \leqslant 2n + 1$. *Then* $\Gamma(p) = 0 \Rightarrow \gamma(p) = 0 \Rightarrow$ *there exists a section of p extending z.*

PROOF. (I) If $g$ is an extension of $z$, the homotopy $\{u_t\}$ can obviously be extended to a homotopy of $s_0$ with $s_1$, hence $\sigma[s_1] = [s_0] = 0$. The second implication is obvious. (II) If $\Gamma(p) = 0$, then $\gamma(p) = 0$ by 2.4.1. Thus, there is a homotopy between $s_0$ and $s_1$, extending $\{u_t\}$, hence a section of $PSp$ extending $iz$. By 2.4.1, $i_\# : [(X, X_0), (A, A_0); p]^z \to [(X, X_0), (A, A_0); PSp]^{iz}$ is onto, and we are done.

*The single difference.* Suppose now that $\{z_t\}$ is a homotopy of partial sections of $p$ over $(A, A_0)$. Suppose that $g_0$ and $g_1$ are sections of $p$ extending $z_0$ and $z_1$, respectively. Recalling the construction in the proof of 2.3.2, we have $(ig_1) \wedge (ig_0)^{-1}$, a section of $\Omega Pp$. Now that section extends $(iz_1) \wedge (iz_0)^{-1}$ which is homotopic (by continuously varying the first index from 1 to 0) to $(iz_0) \wedge (iz_0)^{-1}$, which in turn is canonically null-homotopic. Thus, using 2.1.5, we can define:

DEFINITION 2.5.6. Let $\delta(g_0, g_1; \{z_t\}) \in [(X, X_0), (A, A_0); \Omega Pp]$, the *primitive single difference* be the element represented by $(ig_1) \wedge (ig_0)^{-1}$. If $z_t = z$ for all $t$, write $\delta(g_0, g_1; z)$.

DEFINITION 2.5.7. Let $\Delta(g_0, g_1; \{z_t\}) \in H^0((X, X_0), (A, A_0); p)$, the *single difference class* be the image of $\delta(g_0, g_1; \{z_t\})$ in the direct limit. Write $\Delta(g_0, g_1; z)$ if $z_t = z$ for all $t$.

THEOREM 2.5.8 (I) *The homotopy* $\{z_t\}$ *can be extended to a homotopy of* $g_0$ *with* $g_1 \Rightarrow \delta(g_0, g_1; \{z_t\}) = 0 \Rightarrow \Delta(g_0, g_1; \{z_t\}) = 0$. (II) *Suppose that p is n-connected for some n, and* $\dim(X/A) \leqslant 2n + 1$. *Then* $\Delta(g_0, g_1; \{z_t\}) = 0 \Rightarrow$ $\delta(g_0, g_1; \{z_t\}) = 0 \Rightarrow$ *there exists a homotopy of* $g_0$ *with* $g_1$ *which extends* $\{z_t\}$.

PROOF. Similar to that of 2.5.5.

2.6. *Obstructions to embedding and isotopy.* Much of the following material is from [7]. If $M$ is any manifold, of dimension $n$, let $PM$ be the projective bundle associated with the tangent bundle of $M$, let $R^*M = (M^2 - \Delta M)/T$, where $T(x, y) = (y, x)$, and let $\mathbf{R}^*M = R^*M \cup PM$, a manifold with boundary $PM$. Let $(^N\mathbf{R}^*M, ^NPM) = (\mathbf{R}^*(M \times R^N), P(M \times R^N))$, and let $(^\infty\mathbf{R}^*M, ^\infty PM)$ be the obvious union with the weak topology. The inclusion $(\mathbf{R}^*M, PM) \subset (^\infty\mathbf{R}^*M, ^\infty PM)$ we replace by a fibration of pairs in a standard manner: Let $Y_M$ be the space of all paths $\sigma: I \to ^\infty\mathbf{R}^*M$ such that $\sigma(1) \in \mathbf{R}^*M$ and let $Z_M$ be the space of all paths $\sigma: I \to ^\infty PM$ such that $\sigma(1) \in PM$, and let $\zeta_M: (Y_M, Z_M) \to (^\infty\mathbf{R}^*M, ^\infty PM)$ be evaluation at 0, an $(n - 2)$-connected pair fibration.

Now let $V$ be a compact manifold of dimension $k$, and let $f: V \to M$ be a differentiable map. Choose, once and for all, an embedding $\mathbf{i}: V \subset R^\infty$. Let

$F = \mathbf{R}^*(f, \mathbf{i})$: $(\mathbf{R}^*V, PV) \to ({}^{\infty}\mathbf{R}^*M, {}^{\infty}PM)$. We now have a diagram (basically diagram (3.2-1) of [7], with slightly changed notation):

$$(Y_M, Z_M)$$

(2.6-1)
$$\downarrow \zeta_M$$

$$(\mathbf{R}^*V, PV) \;\overset{F}{\to}\; ({}^{\infty}\mathbf{R}^*M, {}^{\infty}PM)$$

Now define a function $\phi$: $[V \subset M]_f \to [(\mathbf{R}^*V, PV); \zeta_M]_F$ as follows. If $\{f_t\}$ is an embedding homotopy of $f$, i.e., $f_0 = f$ and $f_1$ is an embedding, let $\Phi\{f_t\}$: $(\mathbf{R}^*V, PV) \to (Y_M, Z_M)$ be defined as follows. For any $0 < u < 1$ and any $\mathbf{r} \in \mathbf{R}^*V$, $\Phi\{f_t\}(\mathbf{r})(u) = \mathbf{R}^*(f_u, (1 - u)\mathbf{i})(\mathbf{r})$. Simply then let $\phi[f_t]$ be the homotopy class containing $\Phi\{f_t\}$. It follows directly[2] from Theorems 3.3.1 and 3.3.2 of [7] that:

THEOREM 2.6.1. (I) *If* $2n \geqslant 3(k + 1)$, $\phi$ *is onto.* (II) *If* $2n > 3(k + 1)$, $\phi$ *is one-to-one.*

DEFINITION 2.6.2. For any integer $i$, let $H^i(f) = H^i((\mathbf{R}^*V, PV); F^{-1}\zeta_M)$.

DEFINITION 2.6.3. Let $\Gamma(f) = \Gamma(F) \in H^1(f)$, the *single obstruction* to homotopy of $f$ with an embedding.

DEFINITION 2.6.4. Suppose that $\{f_t^1\}$ and $\{f_t^2\}$ are both $e$-homotopies of $f$. Then let $\Delta(\{f_t^1\}, \{f_t^2\}) = \Delta(\Phi\{f_t^1\}, \Phi\{f_t^2\}) \in H^0(f)$.

Now, from 2.5.2, 2.5.5, and 2.5.8, we immediately have:

THEOREM 2.6.5. *If* $2n > 3(k + 1)$, $[V \subset M]_f$ *is an Abelian affine group, and, if nonempty, it has action group* $H^0(f)$.

THEOREM 2.6.6. *If* $f$ *is homotopic to an embedding,* $\Gamma(f) = 0$. *The converse holds if* $2n \geqslant 3(k + 1)$.

THEOREM 2.6.7. *Suppose* $\{f_t^1\}$ *and* $\{f_t^2\}$ *are embedding homotopies of* $f$. *If they are isotopic,* $\Delta(\Phi\{f_t^1\}, \Phi\{f_t^2\}) = 0$. *The converse holds if* $2n \geqslant 3(k + 1)$.

Finally, the following remarks will show how $[V \subset M]_f$, up to isomorphism, depends only on the homotopy class of $f$. Let $f_t$: $V \to M$, $0 < t < 1$, be any homotopy, where $f_0$ and $f_1$ are differentiable. Let $\{f_t\}^*$: $[V \subset M]_{f_1} \to [V \subset M]_{f_0}$ be the function, where, if $\{g_t\}$ is an $e$-homotopy of $f_1$, $\{f_t\}^*[g_t] = [h_t]$, where $h_t = f_{2t}$ or $g_{2t-1}$, depending on the value of $t$. Clearly $\{f_t\}^*$ is one-to-one and onto, since $\{f_{1-t}\}^*$ is its two-sided inverse.

THEOREM 2.7.1. *If* $2n > 3(k + 1)$, $\{f_t\}^*$ *is an isomorphism of affine groups.*

PROOF. Applying the polyhedral covering homotopy property (pair version)

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[2]Theorems 3.3.1 and 3.3.2 of [7] are correct as stated in that paper, but the proofs are invalid, as has been kindly pointed out by the referree of this paper. A correction is contained here, as an appendix, §8.

of $\zeta_M$, one may easily show that, for any $0 \leqslant u \leqslant 1$, $(i_u)^*$: $[(\mathbf{R}^*V \times I, PV \times I);\ \zeta_M]_{Fp} \to [(\mathbf{R}^*V, PV);\ \zeta_M]_F$ is an isomorphism, where $i_u(\mathbf{r}) = (\mathbf{r}, u)$ for all $\mathbf{r} \in \mathbf{R}^*V$ and $p(\mathbf{r}, u) = \mathbf{r}$. Very simply, checking definitions, one can see that $\{f_i\}^* = (i_0)^*((i_1)^*)^{-1}$, and we are done.

In a similar manner, we shall write $\{f_i\}^*$: $H^i(f_1) \cong H^i(f_0)$ for any integer $i$. We leave the details to the reader.

**3. A spectral sequence.** In this section, let $((X, X_0), (A, A_0))$ be a fixed C.W. pair of pairs, and let $\zeta$: $(E, E_0) \to (X, X_0)$ be a pair fibration. We consider the problem of enumeration of sections of $\zeta$ from a spectral sequence viewpoint.

All pair fibrations shall be over $(X, X_0)$.

**3.1.** *Homotopy sheaves.* Let $\zeta$ have a section $s$. We define $\pi_k(\zeta, s)$, the $k$th *homotopy sheaf* of $\zeta$, to be the sheaf over $X$ determined by a presheaf $\Pi$, where, if $U \subset X$ is open and $U_0 = U \cap X_0$, $\Pi(U) = [(U, U_0);\ \Omega^k\zeta]$. Hence, for any $x \in X$,

$$\pi_k(\zeta, s)_x = \begin{cases} \pi_k(E_x, sx) & \text{if } x \in X - X_0, \\ \pi_k((E_0)_x, sx) & \text{if } x \in X_0. \end{cases}$$

The total space of $\pi_k(\zeta, s)$ need not be Hausdorff.

If each $E_x$ and each $(E_0)_x$ is $k$-simple, $\pi_k(\zeta)$ can be defined regardless of choice (or even existence) of a section. We leave the details to the reader. (Hint: it is sufficient to define $\Pi(U)$ for $U$ contractible.)

Note that the concept of homotopy sheaf is simply a generalization of the usual local system of coefficients for a fibration.

Without any restrictions on $\zeta$, we can always define $\pi_k^S(\zeta) = \text{Lim}_{N \to \infty} \pi_k(\Omega^N S^N\zeta)$, the $k$th *stable homotopy sheaf*.

Let $G$ be a sheaf over $X$. We say that $\zeta$ is an Eilenberg Mac Lane pair fibration of type $(G, n)$ (we will write $\zeta = k(G, n)$) if:

(i) $\zeta$ has a section,

(ii) $\pi_n(\zeta) = G$,

(iii) $\pi_k(\zeta) = 0$ for all $k \neq n$.

It is important to note that $G$ cannot simply be any sheaf of groups, Abelian if $n \geqslant 2$. Only for $G$ satisfying special conditions will $k(G, n)$ exist.

In this context, Theorem 2.5.1 of [7] becomes:

**THEOREM 3.1.1.** *If $\zeta = k(G, n)$ for some sheaf of Abelian groups $G$, then $[(X, X_0), (A, A_0);\ \zeta] \cong H^n(X, A;\ G)$.*

**3.2.** *The fiber of a pair-fibration map.* Let $\zeta'$: $(E', E_0') \to (X, X_0)$ be another pair fibration. We say that $\gamma$ is a pair fibration map, and simply write $\gamma$: $\zeta \to \zeta'$, if $\gamma$: $(E, E_0) \to (E', E_0')$ si simply a fiber-preserving map. Suppose that $\zeta$ has a section $s$. Then $s' = \gamma s$ is a section of $\zeta'$. We define a pair

fibration $\phi[\gamma]: (F, F_0) \to (X, X_0)$, called the *fiber* of $\gamma$, as follows: for each $x \in X$, $F_x$ is the usual homotopy theoretic fiber of $E_x \to E'_x$, that is, $F_x = \{(e, \sigma) | e \in E_x, \sigma: I \to E'_x, \sigma(0) = s'x, \sigma(1) = \gamma e\}$, and $(F_0)_x$ is the homotopy theoretic fiber of $(E_0)_x \to (E'_0)_x$. Let $\lambda: \phi[\gamma] \to \zeta$ be given by $\lambda(e, \sigma) = e$, and let $\iota: \Omega\zeta' \to \phi[\gamma]$ be the obvious inclusion.

Identifying $\Omega\phi[\gamma]$ with $\phi[\Omega\gamma]$ in the obvious way, we have:

LEMMA 3.2.1. *The following long sequence is exact:*

$$\cdots \to \left[(X, X_0), (A, A_0); \Omega\zeta\right] \overset{(\Omega\gamma)_*}{\to} \left[(X, X_0), (A, A_0); \Omega\zeta'\right]$$

$$\overset{\iota_*}{\to} \left[(X, X_0), (A, A_0); \phi[\gamma]\right]$$

$$\overset{\lambda_*}{\to} \left[(X, X_0), (A, A_0); \zeta\right] \overset{\gamma_*}{\to} \left[(X, X_0), (A, A_0); \zeta'\right]$$

PROOF. Since the definition of fiber the natural definition for the category of pair-fibrations over $(X, X_0)$, the details of the proof are routine and obvious.

Similarly, we have:

LEMMA 3.2.2. *The following sequence of homotopy sheaves is exact:*

$$\cdots \overset{\gamma_*}{\to} \pi_k(\Omega\zeta') = \pi_{k-1}(\zeta') \overset{\iota_*}{\to} \pi_k(\phi[\gamma]) \overset{\lambda_*}{\to} \pi_k(\zeta) \overset{\gamma_*}{\to} \pi_k(\zeta') \to \cdots.$$

3.3. *The homotopy killing constructions.*

*The strong topology double mapping cylinder.* Let $A$, $B$, and $C$ be topological spaces, and $\alpha: B \to A$, $\beta: B \to C$ maps. Let $A \cup_\alpha (B \times I) \cup_\beta C$ be the space obtained from $A \cup (B \times I) \cup C$ by identifying $(x, 0)$ with $\alpha x$ and $(x, 1)$ with $\beta x$ for all $x \in B$; but with the strong topology, not the quotient topology. Neighborhoods of points in $B \times (0, 1)$ are as usual in the product topology, while if $a \in A$, a basic neighborhood of $a \in W$ is of the form $U \cup [\alpha^{-1}U \times [0, \varepsilon)]$ for $\varepsilon > 0$ and $U$ a neighborhood of $a$ in $A$. Neighborhoods along $C$ are similar. Equivalently, the strong topology is the strongest topology such that all of the following obvious projections are continuous:

$$A \cup_\alpha (B \times I) \cup_\beta C \to I,$$

$$A \cup_\alpha (B \times [0, 1)) \to A,$$

$$(B \times (0, 1]) \cup_\beta C \to C.$$

$$B \times (0, 1) \to B.$$

Now suppose that $a: (A, A_0) \to (X, X_0)$, $b: (B, B_0) \to (X, X_0)$, $c: (C, C_0) \to (X, X_0)$, $\alpha: (B, B_0) \to (A, A_0)$, and $\beta: (B, B_0) \to (C, C_0)$ are all pair fibrations.

LEMMA 3.3.1. $a \cup_\alpha (b \times I) \cup_\beta c: (A \cup_\alpha (B \times I) \cup_\beta C, A_0 \cup_\alpha (B_0 \times I) \cup_\beta C_0) \to (X, X_0)$ *is a pair fibration.*

PROOF. Omit. See Hall [5] for the proof of a special case, the fiberwise strong topology join.

Now suppose that, for all $n \geqslant 1$, we have pair fibrations $a_n$, $b_n$, $c_n$, $\alpha_n$, and $\beta_n$ (as in the hypotheses of 3.3.1), such that $a_{n+1} = a_n \cup_{\alpha_n} (b_n \times I) \cup_{\beta_n} c_n$ for all $n$. Let $a_\infty = \cup a_n$, with the weak topology.

LEMMA 3.3.2. $a_\infty$ *is a pair fibration.*

PROOF. Essentially by induction on $n$, mimicking the proof of 3.3.1 at each step. We omit the details.

Let $E_X^{S^n} \to X$ be the fibration where, for each $x \in X$, $(E_X^{S^n})_x = (E_x)^{S^n}$, the space of maps $S^n \to E_x$, with the compact open topology. Let $e: E_X^{S^n} \times S^n \to E$ be the evaluation map, and let $K_X^n E = E \cup_e (E_X^{S^n} \times B^{n+1})$, with the strong topology. (Define as follows: Since $B^{n+1}$ is the cone over $S^n$, $K_X^n E = E \cup_e (E_X^{S^n} \times S^n \times I) \cup_\pi E_X^{S^n}$, the strong topology double mapping cylinder, where $\pi$ is projection.) Now let $K^n \zeta: (K_X^n E, K_{X_0}^n E_0) \to (X, X_0)$; by 3.2.1, a pair fibration.

THEOREM 3.3.3. (I) *If* $k < n$, $i_\#: \pi_k(\zeta) \cong \pi_k(K^n\zeta)$. (II) $\pi_n(K^n\zeta) = 0$.

PROOF. We first need a lemma.

LEMMA 3.3.4. *If* $Z$ *is any simplicial complex of dimension less than or equal to* $n$, *and if* $Y$ *is any space, then* $[Z; Y] \to [Z; Y \cup_e (Y^{S^n} \times B^{n+1})]$ *is onto.*

PROOF OF LEMMA. Let $* \in B^{n+1}$ be its center. If $f: Z \to Y \cup_e (Y^{S^n} \times B^{n+1})$, $f$ can be deformed slightly so that its image does not intersect $Y^{S^n} \times \{*\}$. The complement of that subset collapses to $Y$, and we are done.

Returning to the proof of 3.3.3, we have immediately from the lemma that $\pi_k(\zeta) \to \pi_k(K^n\zeta)$ is onto for $k \leqslant n$ and one-to-one for $k < n$. Finally, $\pi_n(\zeta) \to \pi_n(K^n\zeta)$ is the zero map, since the construction attaches an $(n + 1)$-cell to every possible map of $S^n$ to every fiber.

DEFINITION 3.3.5. For $0 \leqslant n \leqslant m$, let $K^{n,m}\zeta$ be defined inductively by $K^{n,m}\zeta = K^n\zeta$, and $K^{n,m+1}\zeta = K^{m+1}(K^{n,m}\zeta)$. Finally, let $\mathbf{K}^n\zeta = \cup_m K^{n,m}\zeta$.

THEOREM 3.3.6. (I) *If* $k < n$, $\pi_k(\zeta) \to \pi_k(\mathbf{K}^n\zeta)$. (II) *If* $k \geqslant n$, $\pi_k(\mathbf{K}^n\zeta) = 0$.

PROOF. Direct from 3.3.3.

3.4. *The spectral sequence determined by a resolution.* Suppose that $\zeta$ has a section. We say that the following commutative diagram of sectioned pair fibrations and maps

(3.4-1)

$$\cdots \xrightarrow{\alpha_3} \zeta_2 \xrightarrow{\alpha_2} \zeta_1 \xrightarrow{\alpha_1} \zeta_0$$

is a *resolution* if, for each integer $n$, there exists an integer $N(n)$ such that $(\alpha_m)_\#\colon \pi_k(\zeta_m) \cong \pi_k(\zeta_{m-1})$ and $(\beta_m)_\#\colon \pi_k(\zeta) \cong \pi_k(\zeta_m)$ for all $k < n$ and all $m > N(n)$. We say that it is a *Postnikov resolution* if, for every $n > 0$, $(\beta_n)_\#\colon \pi_k(\zeta) \cong \pi_k(\zeta_n)$ for all $k < n$, and $\pi_k(\zeta_n) = 0$ for all $k > n$.

**THEOREM 3.4.1.** *If $\zeta$ has a section, $\zeta$ has a Postnikov resolution. Furthermore, if $\gamma\colon \zeta \to \zeta'$, $\gamma$ induces a natural map of Postnikov resolutions.*

**PROOF.** For every integer $n > 0$, let $\zeta_n = \mathbf{K}^{n+1}\zeta$. Let the $\alpha_i$ and $\beta_i$ be the inclusions. By 3.3.6, we are done.

**LEMMA 3.4.2.** *If (3.4-1) is a Postnikov resolution, $\phi[\alpha_n] = k(\pi_n(\zeta), n)$.*

**PROOF.** Directly from 3.2.2.

Suppose diagram (3.4-1) is given. We define (taking $\zeta_q\colon (X, X_0) \to (X, X_0)$ for $q < 0$):

$$E_2^{p,q} = \left[(X, X_0), (A, A_0); \Omega^{-p}\phi[\alpha_q]\right] \quad \text{for } p \le 0,$$

$$D_2^{p,q} = \left[(X, X_0), (A, A_0); \Omega^{-p}\zeta_q\right] \quad \text{for } p \le 0,$$

$$i_2 = (\Omega^{-p}\alpha_q)_\#\colon D_2^{p,q} \to D_2^{p,q-1} \quad \text{for } p \le 0,$$

$$j_2 = (\Omega^{-p-1}\iota)_\#\colon D_2^{p,q} \to E_2^{p+1,q+1} \quad \text{for } p \le -1,$$

$$k_2 = (\Omega^{-p}\lambda)_\#\colon E_2^{p,q} \to D_2^{p,q} \quad \text{for } p \le 0.$$

By 3.2.1, we now have a bigraded exact couple (although there is an edge problem, since $p$ must not be positive). Furthermore, from 3.1.1 and 3.4.2, we immediately have:

**REMARK 3.4.3.** *If (3.4-1) is a Postnikov resolution, $E_2^{p,q} = H^{p+q}(X, A; \pi_q(\zeta))$.*

Let $E_r^{p,q}$, $D_r^{p,q}$, $E_\infty^{p,q}$, and $D_\infty^{p,q}$ be obtained in the usual manner.

**THEOREM 3.4.4.** *If $p \le -1$, the $E_\infty^{p,q}$ (for various $q$) give a composition series for $[(X, X_0), (A, A_0); \Omega^{-p}\zeta]$, provided $\dim(X/A) < \infty$.*

**PROOF.** Let $G^{p,q}$ be the kernel of $(\Omega^{-p}\beta_q)_\#\colon [(X, X_0), (A, A_0); \Omega^{-p}\zeta] \to D_2^{p,q}$. Thus, $G^{p,-1} = [(X, X_0), (A, A_0); \Omega^{-p}\zeta]$ and, for dimensional reasons, $G^{p,q} = 0$ for $q$ sufficiently large. By standard spectral sequence arguments, $E_\infty^{p,q} = G^{p,q-1}/G^{p,q}$, and we are done.

If $\zeta$ is deloopable, i.e., $\zeta = \Omega\eta$, we can, by constructing a Postnikov resolution for $\eta$, obtain a spectral sequence converging to a composition series for $[(X, X_0), (A, A_0); \zeta]$. If $\zeta$ is infinitely deloopable, the exact couple can be constructed with no restrictions on the indices. Since computation of $H^i((X, X_0), (A, A_0); \zeta)$ for $\dim(X/A) < \infty$ involves classifying sections of $\Omega^N S^{i+N}\zeta$ for large $N$, that cohomology theory can also be obtained by a

spectral sequence, where $E_2^{p,q} = H^{p+q}(X, A; \pi_q^S(\zeta))$. We leave the details to the reader.

**4. A geometric interpretation of the affine structure.** Let $f: V \to M$ be a map, where $V$ is a compact manifold of dimension $k$, and $M$ is a manifold of dimension $n$, where $2n > 3(k + 1)$. We give a geometric interpretation of the affine group structure of $[V \subset M]_f$.

By Theorem 2.7.1, it is sufficient to consider the case where $f$ is actually an embedding. We then consider $[V \subset M]_f$ to be an actual Abelian group with identity $[f]$, represented by the constant homotopy.

Let $\{f_t^1\}$, $\{f_t^2\}$ be $e$-homotopies of $f$ such that

(i) $f_t^1$ and $f_t^2$ are differentiable for all $t \in I$.

(ii) For some disjoint closed sets $K_1, K_2 \subset V$, and for all $t \in I$, $f_t^i$ agrees with $f$ on $V - K_i$.

(iii) For any $x_1 \in K_1$, $x_2 \in K_2$, $x_3 \in V - (K_1 \cup K_2)$, and for any $t, u \in I$; $f_t^1(x_1), f_u^2(x_2)$, and $f(x_3)$ are distinct.

Now let $f_t^3: V \to M$ be the $e$-homotopy of $f$ such that, for all $x \in V, t \in I$:

$$f_t^3(x) = \begin{cases} f_t^1(x) & \text{if } x \in K_1, \\ f_t^2(x) & \text{if } x \in K_2, \\ f(x) & \text{otherwise.} \end{cases}$$

THEOREM 4.1. $[f_t^1] + [f_t^2] = [f_t^3]$.

PROOF. We need only show that

$$\left[\iota\Phi\{f_t^1\}\right]\left[\iota\Phi\{f\}\right]^{-1}\left[\iota\Phi\{f_t^2\}\right] = \left[\iota\Phi\{f_t^3\}\right] \in \left[(\mathbf{R}^*V, PV); PS\zeta_M\right]_F,$$

where $\iota: \zeta_M \subset PS\zeta_M$. Pick disjoint closed sets $L_1, L_2$ in $V$ such that $K_i \subset$ Int $L_i$. Choose continuous functions $\rho_1, \rho_2: V \to I$ such that $\rho_i(x) = 1$ for all $x \in K_i$, and $0$ for all $x \notin L_i$. For each $i = 1, 2,$ or $3$, let $\Phi_{i,u}, 0 \leq u \leq 1$, be the homotopy of liftings of $F$ to $\zeta_M$ such that, for all $u$, $\Phi_{i,u}$ agrees with $\Phi\{f_t^i\}$ on $PV$, and such that for all $0 \leq t, u \leq 1$ and all $[x, y] \in R^*V$,

$$\Phi_{i,u}[x, y](t) = \left[\left(f_v^i(x), (1 - t)\mathrm{i}(x)\right), \left(f_w^i(y), (1 - t)\mathrm{i}(y)\right)\right]$$

$$\in R^*(M \times R^\infty) = {}^\infty R^*M,$$

where:

$$v = t\big(u\rho_i(x) + 1 - u\big) \quad \text{and} \quad w = t\big(u\rho_i(y) + 1 - u\big) \quad \text{if } i = 1 \text{ or } 2,$$

$$v = t\big(u\rho_j(x) + 1 - u\big) \quad \text{and} \quad w = t\big(u\rho_j(y) + 1 - u\big)$$

$$\text{if } i = 3, j = 1 \text{ or } 2, \text{ and } \rho_j(x)\rho_j(y) > 0,$$

$$v = w = t(1 - u) \quad \text{if } i = 3, \text{ and } \rho_1(x)\rho_1(y) + \rho_2(x)\rho_2(y) = 0.$$

Note then that $\Phi_0^i = \Phi\{f_t^i\}$ for all $i = 1, 2$, or $3$, and by 2.3.4, letting $D_i = Q(L_i^2) \subset \mathbf{R}*V$ for $i = 1$ or $2$, we obtain: $[\iota\Phi_1^1][\iota\Phi\{f\}]^{-1}[\iota\Phi_1^2] = [\iota\Phi_1^3]$, and we are done.

Using a general position argument, based on the dimensional restriction, it is possible to show that any two $e$-homotopies of $f$ are respectively isotopic to a pair of $e$-homotopies satisfying (i) through (iii) above. Thus Theorem 4.1 suffices to describe the affine structure on $[V \subset M]_f$.

In later calculations, the following result is useful:

THEOREM 4.2. *If $f': V \to M$ is another map such that $f|D$ is homotopic to $f'|D$ for some $D \subset V$ such that $V - D \cong R^n$, then $[V \subset M]_f \cong [V \subset M]_{f'}$, if both are nonempty.*

PROOF. Without loss of generality, $f$ and $f'$ are both embeddings, and $f'|D = f|D$. If $\alpha \in [V \subset M]_f$, we may, by a general position argument, choose an $e$-homotopy $\{f_t\}$ of $f$ such that

(i) $[f_t] = \alpha$,

(ii) $f_t|(V - D) = f|(V - D)$ for all $t$,

(iii) $f_t(D) \cap f'(V - D)$ is empty for all $t$.

Now let $\phi(\alpha) = [f_t']$, where, for all $0 \leqslant t \leqslant 1$, $f_t'(x) = f_t(x)$ if $x \in D$, $f_t'(x) = f'(x)$ if $x \in (V - D)$. Using 4.1 and standard general position arguments, one may easily show that $\phi: [V \subset M]_f \to [V \subset M]_{f'}$ is well defined, and is an isomorphism.

5. **The structure of $\pi_n \zeta_M$.** Let $M$ be any connected $n$-manifold. In this section, we explicitly compute the sheaf $\pi_n \zeta_M$ over $^\infty\mathbf{R}*M$, provided $n \geqslant 6$. (The sheaf $\pi_{n-1}\zeta_M$ was computed in [7], provided $n \geqslant 5$; we restate the results here. For $k < n - 1$, $\pi_k \zeta_M = 0$.)

5.1. *Building a twisted sheaf.* Suppose that $X$ is any path-connected space with basepoint $x_0$, and $S$ is a sheaf over $X$ with a local product structure. Let $S_x$ be the stalk of $S$ over any $x \in X$. If $a \in \pi_1 X$, let $\alpha: (I, \partial I) \to (X, x_0)$ be a loop representing $a$, and let $\tilde\alpha: I \times S_{x_0} \to S$ be a map such that $\tilde\alpha(t, s) \in S_{\alpha(t)}$ and $\tilde\alpha(1, s) = s$ for all $t \in I$, $s \in S_{x_0}$. Let $\langle a \rangle: S_{x_0} \to S_{x_0}$ be the automorphism where $\langle a \rangle(s) = \tilde\alpha(0, s)$. Let $\mu_S: \pi_1 X \times S_{x_0} \to S_{x_0}$ be the left action where $\mu_S(a, s) = \langle a \rangle(s)$ for all $a \in \pi_1 X$, $s \in S_{x_0}$.

LEMMA 5.1.1. *Let $X$ be path connected and locally simply connected, with basepoint. Let $G$ be any group and let $\mu: \pi_1 X \times G \to G$ be any left action. Then there exists a sheaf $S = S(G, \mu)$, with local product structure over $X$, such that $S_{x_0} = G$ and $\mu_S = \mu$. Furthermore, $S$ is unique with these properties, up to isomorphism.*

PROOF. Let $\pi: Y \to X$ be the universal covering of $X$, and let $\nu: \pi_1 X \times Y \to Y$ be the associated (left) action, where $\pi y_1 = y_2$ if and only if $\nu(a, y_2)$ for

some $a \in \pi_1 X$. Then let $S(G, \mu)$ be the quotient space of $Y \times G$ obtained by identifying each $(y, g)$ with $(\nu(a, y), \mu(a, g))$ for all $a \in \pi_1 X$. Trivially, the required conditions are satisfied. The proof of uniqueness is trivial and straightforward, hence we omit it.

5.2. *Preliminary definitions.* For any topological space $X$, let $\Gamma X$ be the quotient space of $X \times X \times S^\infty$ obtained by identifying each $(x, y, a)$ with $T(x, y, a) = (y, x, -a)$. If $f: X_1 \to X_2$ is any map, let $\Gamma f: \Gamma X_1 \to \Gamma X_2$ be the obvious map. Note that if $*$ is a single point space, $\Gamma* = p^\infty = S^\infty / T$. Let $\Gamma 0: \Gamma X \to P^\infty$, where $0: X \to *$ is the collapsing map. If $* \in X$ is a basepoint, let $j = \Gamma i: P^\infty \to \Gamma X$, where $i: * \to X$ is the inclusion. Let $\pi: X \times X \times S^\infty \to \Gamma X$ denote the 2-1 covering, and, by a slight abuse of notation, let $\pi: X \times X \to \Gamma X$ denote the (equivalent to the covering) inclusion, where $\pi(x, y) = [x, y, *]$, $* \in S^\infty$ a basepoint. We may consider $X \times P^\infty \subset \Gamma X$; if $x \in X$ and $a \in S^\infty$, identify $(x, [a])$ with $[x, x, a]$.

Let $X$ be any topological space with basepoint, and let $\pi_k = \pi_k X$, written multiplicatively if $k = 1$, additively if $k > 1$. Let $\gamma_k: \pi_1 \times \pi_k \to \pi_k$ be the usual left action, determined by the map $\gamma_k: S^k \to S^1 \vee S^k$, where

$$\gamma_k[x_1, x_2, \ldots, x_k] = \begin{cases} [2x_1] \in S^1 & \text{if } 0 \leqslant x_1 \leqslant \tfrac{1}{2}, \\ [2x_1 - 1, x_2, \ldots, x_k] \in S^k & \text{if } \tfrac{1}{2} \leqslant x_1 \leqslant 1, \end{cases}$$

where $S^k = I^k / \partial I^k$. We let $x^a = \gamma_k(a, x)$ for all $a \in \pi_1$, $x \in \pi_k$. Let $\alpha_k: \pi_2 \otimes \pi_k \to \pi_{k+1}$, for $k \geqslant 2$, be the Whitehead product, determined by the map $\alpha_k: S^{k+1} \to S^2 \vee S^k$, where

$$\alpha_k(x_1, x_2, \ldots, x_{k+2}) = \begin{cases} [x_1, x_2] \in S^2 & \\ & \text{if } x_i \in \partial I \text{ for some } 3 \leqslant i \leqslant k + 2, \\ [x_3, \ldots, x_{k+2}] \in S^k & \text{if } x_i \in \partial I \text{ for } i = 1 \text{ or } 2, \end{cases}$$

where $S^2 = I^2 / \partial I^2$, $S^k = I^k / \partial I^k$, and $S^{k+1} = \partial I^{k+2}$.

If $X$ is a manifold and $x \in \pi_k$, let $w_k x \in Z_2$ be the value of $f^* w_k X \in H^k(S^k; Z_2)$ (the Stiefel-Whitney class), where $f: S^k \to X$ represents $x$. If $a \in \pi_1$, let $(-1)^a = 1$ if $w_1 a = 0$, $-1$ otherwise.

Again, if $X$ is a manifold, and if $a \in \pi_1$, $a^2 = 1$ (the identity), let $f_a: P^2 \to X$ be a map which sends the generator of $\pi_1 P^2$ to $a$, and elt $\tilde{a} = [f_a s] \in \pi_2$, where $s: S^2 \to P^2$ is the covering map onto the real projective plane. Let $k_a$ be an integer such that the vector bundles $f_a^{-1}\tau$ and $k_a h$ are stably equivalent, where $\tau$ is the tangent bundle of $X$ and $h$ is the canonical line bundle over $P^2$. Note that the *pair* $(k_a, \tilde{a})$ has indeterminacy (because of the choice of $f_a$) $(4Z \oplus 0) + \text{Im } \chi_a$, where $\chi_a x = (mw_2 x, x + x^a)$ for all $x \in \pi_2$, where $m: Z_2 \to Z_4$ is the monomorphism. (Hint: recall that $\tilde{K}^0(P^2) = Z_4$.)

Let $G$ and $N$ be groups, and let $\varphi: G \to \text{Aut } N$ be any homomorphism. Let

$N \times_\varphi G$ be the semidirect product. Specifically, as a set, $N \times_\varphi G$ is the Cartesian product $N \times G$, with the operation $(n_1, g_1)(n_2, g_2) = (n_1\varphi(g_1)n_2, g_1 g_2)$. Note that if $\varphi$ is trivial, $N \times_\varphi G$ is simply the direct sum. We always have $G \subset N \times_\varphi G$ and $N \lhd N \times_\varphi G$, where we identify $N$ and $G$ with $N \times \{1\}$ and $\{1\} \times G$, respectively.

For any (multiplicatively written) group $G$, let $ZG$ be the group ring of $G$, which we represent as finite formal sums of elements of $G$. Thus, $G \subset ZG$.

5.3. *Structure of* $\pi_{n-1}\zeta_M$ *and* $\pi_n\zeta_M$. Now let $X = {}^\infty R^* M$, and $X_0 = {}^\infty PM$. Once and for all, fix basepoints $* \in M$ and $* \in SM$ over $*$, and let $* \in PM$ also denote the image of $*$. Let $\pi_k = \pi_k M$, written multiplicatively if $k = 1$, additively otherwise. For consistency of notation, we introduce a group $T_2 = \{1, m\}$, a multiplicative group of two elements. Let $H_2 = \{0, \eta\} \cong Z_2$, the stable 1-stem in the homotopy of spheres, which we treat as a $Z_2$-module. For any group $G$, let $\Delta G \subset G \oplus G$ be the diagonal.

LEMMA 5.3.1. (I) $\pi_1 X = (\pi_1 \oplus \pi_1) \times_\varphi T_2$, *where* $\varphi(m)(a, b) = (b, a)$ *for all* $a, b \in \pi_1$. (II) $\pi_1 X_0 = \Delta\pi_1 \oplus T_2$, *and* $i: X_0 \subset X$ *induces the inclusion of groups described in 5.2.*

PROOF. Let $\iota: M \times R^\infty \to R^\infty$ be any embedding, and let $\rho: RR^\infty \to S^\infty$ be the equivalent retraction, determined by $\rho(u, v) = \|u - v\|^{-1}(u - v)$ for all $(u, v) \in RR^\infty$. Let $\psi = (p_1^2 Q, \rho R\iota): (R(M \times R^\infty), S(M \times R^\infty)) \to (M \times M \times S^\infty, \Delta M \times S^\infty)$ which is equivariantly a homotopy equivalence of pairs, by an elementary obstruction theory argument. Passing to quotient spaces under the involution $T$, we have that the pair $(X, X_0)$ is of the homotopy type of $(\Gamma M, M \times P^\infty)$.

We have a partially split exact sequence:

$$1 \to \quad \pi_1 M^2 \quad \to \quad \pi_1 \Gamma M \quad \overset{c_\#}{\underset{j_\#}{\rightleftarrows}} \quad \pi_1 P^\infty \quad \to 1$$
$$\| \qquad\qquad\qquad \| \qquad\qquad\qquad \|$$
$$\pi_1 \oplus \pi_1 \qquad\quad \pi_1 X \qquad\qquad T_2$$

A simple deck-transformation argument then verifies that $\pi_1 M$ is the desired semi-direct product. (II) follows trivially.

Now let $\theta_M: Y \to X$ and $\rho_M: Z \to X_0$ be the fibrations which (as functions) agree with the pair fibration $\zeta_M$.

LEMMA 5.3.2. (I) *If* $n > 5$, $\pi_{n-1}\theta_M = S(Z\pi_1, \mu')$, *where*
(i) $\mu'((b, c, 1), a) = (-1)^c bac^{-1}$,
(ii) $\mu'((b, c, m), a) = (-1)^n(-1)^{ac}ba^{-1}c^{-1}$ *for all* $a, b, c \in \pi_1$.
(II) *If* $n > 6$, $\pi_n\theta_M = S(Z\pi_1 \otimes (\pi_2 \oplus H_2), \mu)$, *where*
(i) $\mu((b, c, 1), a \otimes (x, \lambda\eta)) = (-1)^c bac^{-1} \otimes (x^c, \lambda\eta)$,
(ii) $\mu((b, c, m), a \otimes (x, \lambda\eta)) = (-1)^{n+1}(-1)^{ac}ba^{-1}c^{-1} \otimes (x^{c^{-1}a}, (\lambda + w_2 x)\eta)$ *for all* $a, b, c \in \pi_1, x \in \pi_2$, *and* $\lambda \in Z_2$.

We postpone the proof.

Identify $Z$ with $Z\{1\} \subset Z\pi_1$. Let $\nu': ((Z\Delta\pi_1) \oplus T_2) \times Z \to Z$ and $\nu: ((Z\Delta\pi_1) \oplus T_2) \times (Z \otimes H_2) \to Z \otimes H_2$ be the restrictions of $\mu'$ and $\mu$, respectively.

LEMMA 5.3.3. (I) *If* $n \geqslant 5$, $\pi_{n-1}\rho_M = S(Z, \nu')$. (II) *If* $n \geqslant 6$, $\pi_n\rho_M = S(Z \otimes H_2, \nu)$.

We leave the proof to the reader. (Hint: the fiber of $\rho_M$ is $S^{n-1}$.)

One final lemma completes the description of $\pi_{n-1}\zeta_M$ and $\pi_n\zeta_M$.

LEMMA 5.3.4. *If* $\pi_k\rho_M$ *is a subsheaf of* $(\pi_k\theta_M)|X_0$, *then* $\pi_k\zeta_M$ *is the subsheaf of* $\pi_k\theta_M$ *whose stalks over* $X_0$ *agree with those of* $\pi_k\theta_M$, *and whose stalks over* $X - X_0$ *agree with those of* $\pi_k\rho_M$.

PROOF. Follows directly from the definition of $\pi_k\zeta_M$.

5.4. *Proof of Lemma* 5.3.2. Part (I) of 5.3.2 is simply Lemmas 3.4.2, 3.4.4, and 3.4.5 of [7]. (However, we have replaced the right actions of that paper by left actions.)

Now we saw in the proof of 3.4.2 of [7] that the fiber of $\theta_M$ is the fiber $\Phi$ of the inclusion $\tilde{M}^{00} \subset \tilde{M}$, where $\tilde{M}$ is the universal covering space of $M$, and $\tilde{M}^{00}$ is the universal covering space of $M^0 = M - \{*\}$. A straightforward Serre spectral sequence argument reveals that

(i) $\pi_{n-1}(\Phi) = Z\pi_1$,

(ii) $\pi_n(\Phi) = Z\pi_1 \otimes (\pi_2 \oplus H_2)$,

where composition with $\eta: S^n \to S^{n-1}$ is represented by $\otimes \eta$, and where, for each $a \in \pi_1$ and $x \in \pi_2$, $a \otimes x$ is represented by the map $\phi$ in the following homotopy commutative diagram, where $S^2 \overset{x}{\to} \tilde{M}^{00} \subset \tilde{M} \to M$ represents $x$:

$$
\begin{array}{ccc}
S^n & \overset{\phi}{\to} & \Phi \\
\alpha_{n-1}\downarrow & & \downarrow \\
S^{n-1} \vee S^2 & \overset{a \vee x}{\to} & \tilde{M}^{00} \subset \tilde{M}
\end{array}
$$

The reader can easily verify that (II) need only be checked for the following five special cases (where $a, b, c \in \pi_1$; $x \in \pi_2$):

(i) $((b, 1, 1), a \otimes \eta) = ba \otimes \eta$,

(ii) $((1, 1, m), a \otimes \eta) = a^{-1} \otimes \eta$,

(iii) $((b, 1, 1), a \otimes x) = ba \otimes x$,

(iv) $((1, c, 1), 1 \otimes x) = (-1)^c c^{-1} \otimes x^c$,

(v) $((1, 1, m), 1 \otimes x) = (-1)^{n+1} \otimes ((w_2 x)\eta + x)$.

Now (i), (ii) follow immediately from (I), by composition with $\eta$, while the proof of (iii) involves (I), together with mere straightforward checking of maps.

To prove (iv), let $\gamma: I \to M$ be a loop representing $c$. Let $\psi_t: S^2 \to M$, $0 \leqslant t \leqslant 1$, be a homotopy such that $\psi_t(*) = \gamma(t)$ for all $t$, and $\psi_1$ represents $x$. Then $\psi_0$ represents $x^c$.

We briefly introduce a general construction. If $Y$ is any pointed space, $B \subset Y$, and if $f: (I^k, \partial I^k) \to (Y, B)$ is a map such that $f(*) = *$ for some $* \in I^k$, let $\langle f \rangle: S^{k-1} = \partial I^k \to F$ ($F$ = homotopy theoretic fiber of the inclusion $B \subset Y$) be defined as follows: let $\rho: CS^{k-1} \cong I^k$ ($CX$ = cone over $X$) and let $\langle f \rangle(v) = \sigma_v \in F \subset Y^I$, where $\sigma_v(t) = f(\rho[v, t])$. By a slight modification, it is not necessary to assume $B \subset Y$, only $B \to Y$.

Returning to the proof, choose a homotopy $\theta_t: (I^n, S^{n-1}) \to (\tilde{M}^2, \tilde{R}M)$, where $\tilde{R}M$ is the universal covering space of $RM = M^2 - \Delta M$, such that:

(i) $p_2\theta_t(v) = \gamma(t)$ for all $t \in I$,

(ii) $\langle \theta_1 \rangle$ represents $1 \in Z\pi_1 = \pi_{n-1}(\Phi)$,

where $p_2: \tilde{M}^2 \to \tilde{M}$ is projection to the second factor. Clearly $\langle \theta_0 \rangle$ represents $(-1)^c c^{-1} \in Z\pi_1 = \pi_{n-1}(\Phi)$. Now, for all $t \in I$, we have a homotopy commutative diagram:

$$
\begin{array}{ccccc}
S^n & \xrightarrow{\quad} & H & \xrightarrow{\;h_t\;} & \Phi \\
& \searrow{\alpha_{n-1}} & \downarrow & & \downarrow \\
& & S^2 \vee S^{n-1} \xrightarrow{\xi_t \vee \theta_t} & RM \xrightarrow{\;p^2\;} & RM \\
& & \cap & \cap & \cap \\
& & S^2 \vee I^n \xrightarrow{\xi_t \vee \theta_t} & M^2 \xrightarrow{\;p^2\;} & M^2
\end{array}
$$

where $p: \tilde{M} \to M$ is the covering map, $p^2\xi_t = i_1\psi_t$ for all $t$, $\xi_1(*) = *$, and $\beta$ is chosen independent of $t$. By definition, $[\beta h_1] = 1 \otimes x$ and $\mu((1, c, 1), 1 \otimes x) = [\beta h_0] = (-1)^c(c^{-1} \otimes x^c)$, and (iv) is verified.

To prove (v), let $g: R^n \to M$ be a coordinate patch such that $g(0) = *$. Let $\kappa: (I^n, S^{n-1}) \to (R^n, R^n - \{0\})$ be the obvious orientation preserving equivalence. Then $\langle g\kappa \rangle = \pm 1 \in Z \subset Z\pi_1 = \pi_{n-1}(\Phi)$. We alter $g$ if necessary to insure that $\langle g\kappa \rangle = 1$.

Let $(SM)_*$ be the fiber of $SM \to M$ over $*$. Choose a homeomorphism $\iota: S^{n-1} \to (SM)_*$ such that $\langle \lambda \rangle = 1 \in \pi_{n-1}(\Phi)$, where $\lambda: (I^n, S^{n-1}) \to (M^2, RM)$ is chosen such that $\lambda(I^n) = * \in M^2$, and $\lambda|S^{n-1} = \iota$. When convenient, we shall identify $S^{n-1}$ with $(SM)_*$.

Let $r: E \to S^2$ be the $S^{n-1}$ bundle which is the pullback of $SM \to M$, where $\gamma: S^2 \to M$ classifies $x \in \pi_2$, and let $j: S^2 \to E$ be a section of $r$. Let $e: I^2 \times S^{n-1} \to E$ be a map such that the following diagram commutes:

$$\begin{array}{ccccc}
L \times S^{n-1} & \xrightarrow{\;p_2\;} & S^{n-1} & \xrightarrow{\;\iota\;} & (SM)_\gamma \\
\cap & & \cap & & \cap \\
I^2 \times S^{n-1} & \xrightarrow{\;e\;} & E & \xrightarrow{\;\tilde{\gamma}\;} & SM \\
\;\Big\| p_1 \;\; i_1 & & \;\Big\| j \;\; r & & \Big\downarrow \\
I^2 & \xrightarrow{\;c\;} & S^2 & \xrightarrow{\;\gamma\;} & M
\end{array}$$

where $p_i$ is the projection on the $i$th factor, $c$ collapses $\partial I^2$ to $* \in S^2$, $L = \{1\} \times I \cup I \times \partial I \subset I^2$, and $i_1$ is inclusion along the first factor. Let $y_t\colon I^2 \to M^2, 0 \leqslant t \leqslant 1$, be the homotopy such that

$$y_t(u, v) = \begin{cases} (\gamma[1 - u, v], *) & \text{if } 0 \leqslant u \leqslant t, \\ (\gamma[1 - t, v], \gamma[u - t, v]) & \text{if } t \leqslant u \leqslant 1. \end{cases}$$

Let $W = (S^1 \times S^{n-1}) \cup (I^2 \times \{*\}) \subset I^2 \times S^{n-1}$, where $S^1 = \partial I^2$, and choose a homotopy $\tilde{y}_t\colon W \to \mathbf{R}M, 0 \leqslant t \leqslant 1$, such that $\tilde{y}_t|S^1 \times S^{n-1} = \tilde{\gamma}e(J_t \times 1)$ for all $t$, where $J_t\colon S^1 \to I^2$ is given by $J_t(u, v) = (1 - tu, v)$ for all $(u, v) \in S^1 \subset I^2$; and such that $Q\tilde{y}_t|I^2 \times \{*\}$ and $y_t p_1|I^2 \times \{*\}$ are homotopic, rel $S^1 \times \{*\}$, for all $t$; where $Q\colon \mathbf{R}M \to M^2$ is the usual quotient map.

The following diagram illustrates the next portion of the argument.

$$\begin{array}{ccccc}
S^1 \times S^{n-1} & \xrightarrow{\;J_t\;} & I^2 \times S^{n-1} & \xrightarrow{\;\tilde{\gamma}e\;} & SM \\
\cap & & \tilde{y}_t & & \cap \\
W & & \longrightarrow & & \mathbf{R}M \\
\Big\downarrow p_1 & \nearrow \quad F \xrightarrow{\;z_t\;} \Phi \quad \nearrow & & & \Big\downarrow Q \\
I^2 & & \xrightarrow{\;y_t\;} & & M^2
\end{array}$$

Let $z_t\colon F \to \Phi, 0 \leqslant t \leqslant 1$, be the homotopy induced by $\{y_t\}, \{\tilde{y}_t\}$ (where $F = $ fiber of $p_1\colon W \to I^2$). Let $i_2\colon S^{n-1} \subset W$ be inclusion along the second factor, and let $\omega = (d_\#)^{-1}[\alpha_{n-1}] \in \pi_n W = \pi_n F$, where $d\colon W \to S^2 \vee S^{n-1}$ is the map which collapses $S^1 \times \{*\} \subset W$. Clearly $(z_0)_\# = (-1)^n \mu((1, 1, m), 1 \otimes x)$; it remains to show only that $(z_1)_\# \omega = -(1 \otimes x) + (w_2 x)1 \otimes \eta$.

Note that $\tilde{y}_1\colon S^1 \times \{*\} \to j(*) \in (SM)_* \subset \mathbf{R}M$, thus $y^*\colon S^2 \to \mathbf{R}M$, where $y^*[u, v] = y_1(u, v, *)$ is well defined. We then have a commutative diagram (where $S^2$ is identified with $jS^2 \subset E$):
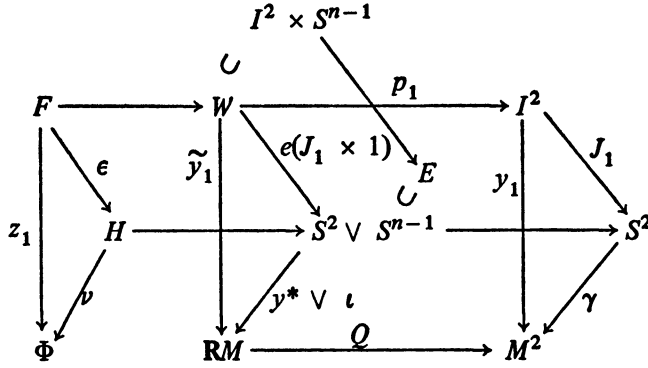
where $\varepsilon, \nu$ are the appropriate maps on the fibers. Now $I^2 \times S^{n-1} = W$ $\cup_w e^{n+1}$, where $w: S^n \to W$ classifies $\omega$, and $E = S^2 \vee S^{n-1} \cup_\lambda e^{n+1}$, where $\lambda = e(J_1 \times 1)w$. Now $\text{Sq}^2\phi = \gamma^* w_2 M \cup \phi$, and $\phi \cup \sigma = - e^{n+1}$ (since $J_1$ has degree $-1$), where $\phi \in H^{n-1}(E; Z)$ is the generator and $\sigma \in H^2(S^2; Z)$ is the fundamental class. Thus $[\lambda] = -[\alpha_{n-1}] + (w_2 x)[\eta i_2] \in \pi_n(S^2 \vee S^{n-1})$, and we are done.

6. **Explicit computation of $[S^k \subset M^{2k}]_f$.** In this section, $[S^k \subset M^n]_f$ is explicitly stated, in terms of generators and relations, where $M$ is any manifold without boundary, $f: S^k \to M$ is any embedding, and $n = 2k + 1$, $n \geqslant 2$, or $n = 2k, k \geqslant 3$.

THEOREM 6.0.1. *If $k \geqslant 2$, $n = 2k + 1$, then $[S^k \subset M^n]_f$ is generated by elements $y_a$ for all $a \in \pi_1 M$, subject only to the following relations:*
    (i) $y_1 = 0$, where $1 \in \pi_1 M$ is the identity;
    (ii) For any $a \in \pi_1 M$, $y_{a^{-1}} = (-1)^{k+1}(-1)^a y_a$.

THEOREM 6.0.2. *If $k \geqslant 3$, $n = 2k$, then $[S^k \subset M^n]_f$ is generated only by elements:*
    (i) $z_a$ for all $a \in \pi_1 M$ such that $a^2 = 1$ and $(-1)^k(-1)^a = 1$.
    (ii) $y_a \otimes u$ for all $a \in \pi_1 M$ and all $u \in \pi_2 M \oplus H_2$, where $H_2 = \{0, \eta\}$ is the stable 1-stem in the homotopy of spheres.
    *And is subject only to the relations:*
    (iii) $y_a \otimes u + y_a \otimes v = y_a \otimes (u + v)$ for any $a \in \pi_1 M, u, v \in \pi_2 M \oplus H_2$.
    (iv) $y_1 \otimes \eta = 0$.
    (v) $z_1 = 0$.
    (vi) $y_a \otimes \eta = y_{a^{-1}} \otimes \eta$ for any $a \in \pi_1 M$.
    (vii) $y_a \otimes x + (-1)^a(-1)^k y_{a^{-1}} \otimes x^a + (w_2 x) y_a \otimes \eta = 0$ for all $a \in \pi_1 M$, $x \in \pi_2 M$.
    (viii)

$$y_a \otimes \tilde{a} + \left(\binom{k-3}{2} + \binom{2k-k_a}{2}\right) y_a \otimes \eta = 0$$

*for all* $a \in \pi_1 M$ *such that* $a^2 = 1$ *and* $(-1)^a (-1)^k = -1$.
  (ix)

$$2z_a = y_a \otimes \tilde{a} + \left( \binom{2k-k_a}{2} + \binom{k-3}{2} + \binom{k}{1} \right) y_a \otimes \eta$$

*for all* $a \in \pi_1 M$ *such that* $a^2 = 1$ *and* $(-1)^a (-1)^k = 1$.

Finally, these representations are natural, i.e.;

THEOREM 6.0.3. *Let* $g: M \subset M'$ *be an embedding, where* $M'$ *is another manifold. Then* (I) *If* $k \geqslant 2$, $n = 2k + 1$, $g_\# y_a = y_{g_* a}$ *for all* $a \in \pi_1 M$. (II) *If* $k \geqslant 3$, $n = 2k$, $g_\#(y_a \otimes x) = y_{g_* a} \otimes g_\# x$ *and* $g_\#(y_a \otimes \eta) = y_{g_* a} \otimes \eta$ *for all* $a \in \pi_1 M$, $x \in \pi_2 M$, *and* $g_\# z_a = z_{g_* a}$ *if* $a \in \pi_1 M$, $a^2 = 1$, *and* $(-1)^a (-1)^k = 1$.

6.0.1 and 6.0.2 are proved below (6.0.1 was incorrectly stated as Theorem 1.2.1 of [7], the result there is off by a sign), while we leave the proof of 6.0.3 to the reader. Basically, 6.0.3 follows from the fact that all of the constructions are natural. Examination of those constructions reveals that $y_a$, $y_a \otimes x$ and $y_a \otimes \eta$ are canonically chosen, while $z_a$ has an indeterminacy, as defined below, the group generated by $y_a \otimes \tilde{a}$, $y_a \otimes \eta$, and $y_a \otimes x + y_a \otimes x^a$ for all $x \in \pi_2 M$. By making a once-and-for-all choice in each of the four universal examples (see the proof of 6.1.3), this indeterminacy could possibly be further reduced.

6.1. *Proof of* 6.0.1 *and* 6.0.2. Without loss of generality, $f$ may be assumed inessential (cf. 4.2). Now, if $k \geqslant 2$, $n = 2k + 1$, there is only one nonzero $E_\infty$ term in the composition series for $[S^k \subset M]_f = H^0(f)$, namely $E_\infty^{0,n-1}$. Now $E_2^{0,n-1} = H^{n-1}(\mathbf{R}^* S^k; F^{-1}\pi_{n-1}\zeta_M)$, and our result follows immediately from Lemma 6.1.1, below, where $y_a$ corresponds to $[a]$ for all $a \in \pi_1 = \pi_1 M$.

Suppose now that $k \geqslant 3$, $n = 2k$. Let $\pi_1 = \pi_1 M$, $\pi_2 = \pi_2 M$. The only $E_2$ terms which play a role in the computation of $[S^k \subset M]_f$ are $E_2^{0,n-1} = H^{n-1}(\mathbf{R}^* S^k; F^{-1}\pi_{n-1} \zeta_M)$, $E_2^{-1,n-1} = H^{n-2}(\mathbf{R}^* S^k; F^{-1}\pi_{n-1}\zeta_M)$, and $E_2^{0,n} = H^n(\mathbf{R}^* S^k; F^{-1}\pi_n \zeta_M)$; and the only relevant differential is $d_2$ is the exact sequence:

(6.1-1)           $E_2^{-1,n-1} \xrightarrow{d_2} E_2^{0,n} \xrightarrow{\lambda} H^0(f) \xrightarrow{\rho} E_2^{0,n-1} \to 0$.

Now if $e: 0 \to A \to B \to C \to 0$ is any short exact sequence of Abelian groups, $e$ determines a homomorphism $\Phi_2: K_2 \to A/2A$, where $K_2 = \{x \in C | 2x = 0\}$, as follows: for $x \in K_2$, $\Phi_2 x = \lambda^{-1} 2\rho^{-1} x$. By Theorem 5.1 of [9], knowledge of $\Phi_2$ suffices to determine $B$ as an extension of $C$ by $A$ if $2C = 0$. By 6.1.1, $2E^{0,n-1} = 0$; then (for the sequence (6.1-1)) the groups are given by 6.1.1, and $d_2$ and $\Phi_2$ are given by 6.1.3, and Theorem 6.0.2 follows immediately; where, for $a \in \pi_1$, $x \in \pi_2$, $\lambda[a \otimes x] = y_a \otimes x$, $\lambda[a \otimes \eta] = y_a \otimes \eta$; and, if $a^2 = 1$ and $(-1)^a (-1)^k = 1$, $\rho z_a = [a]$.

Recall that $H_2 = \{0, \eta\} \cong Z_2$.

For any integer $r$, $k \leqslant r \leqslant 2k$, let $K_{n-1}^r = Z\pi_1$ and $K_n^r = Z\pi_1 \otimes (\pi_2 \oplus H_2)$. For $r < k$ or $r > 2k$, let $K_{n-1}^r = K_n^r = 0$. Define $\delta_{n-1}^r: K_{n-1}^r \to K_{n-1}^{r+1}$ and $\delta_n^r: K_n^r \to K_n^{r+1}$ for all $r$ as follows: for $k \leqslant r < 2k$, and for $a \in \pi_1$, $x \in \pi_2$:

(i) $\delta_{n-1}^r(a) = a + (-1)^{r+k+n+1}(-1)^a a^{-1}$,

(ii) $\delta_n^r(a \otimes x) = a \otimes x + (-1)^a(-1)^{r+k+n}a^{-1} \otimes x^a + (w_2 x)a^{-1} \otimes \eta$,

(iii) $\delta_n^r(a \otimes \eta) = a \otimes \eta + a^{-1} \otimes \eta$.

We now define $H_s^r = \operatorname{Ker} \delta_s^r / \operatorname{Im} \delta_s^r$ for $s = n - 1$ or $n$.

LEMMA 6.1.1. (I) *For all* $r$, *and for* $s = n - 1$ *or* $n$,

$$H^r\left(\mathbf{R}^*S^k, PS^k; F^{-1}\pi_s\zeta_M\right) = H^r\left(\mathbf{R}^*S^k, PS^k; F^{-1}\pi_s\theta_M\right) = H_s^r.$$

(II) *For all* $r > k$, *and for* $s = n - 1$ *or* $n$, $j^*: H_s^r \to H^r(\mathbf{R}S^k; F^{-1}\pi_s\zeta_M)$ *is onto, and*

$$\operatorname{Ker} j^* = \begin{cases} (Z \cap \operatorname{Ker} \delta_{n-1}^r)/\operatorname{Im} \delta_{n-1}^{r-1} & \text{if } s = n - 1, \\ (Z \otimes H_2 \cap \operatorname{Ker} \delta_n^r)/\operatorname{Im} \delta_n^{r-1} & \text{if } s = n. \end{cases}$$

PROOF (I). Trivially,

$$H^*\left(\mathbf{R}^*S^k, PS^k; F^{-1}\pi_s\zeta_M\right) = H^*\left(\mathbf{R}^*S^k, PS^k; F^{-1}\pi_s\theta_M\right),$$

since these two coefficient sheaves differ only on $PS^k$.

We now consider $S^k \subset R^{k+1}$ to be the unit sphere, and $\pi: S^k \to P^k$ the covering map onto real projective $k$-space. For any $x \in P^k$, let $[x] \subset R^{k+1}$ be the line through $0$ determined by $x$. Let $\nu$ be the $k$-plane bundle over $P^k$ such that, for each $x \in P^k$, $\nu_x = [x]^\perp \subset R^{k+1}$. Clearly, $\pi^{-1}\nu = \tau$, the tangent bundle of $S^k$, and $\nu \oplus h = (k + 1)$, the trivial $(k + 1)$-plane bundle, where $h$ is the canonical line bundle over $P^k$ and "$\oplus$" denotes Whitney sum. Thus $w_i\nu = u^i$ for all $i$, where $u \in H^1(P^k; Z_2)$ is the generator.

For any vector bundle $\xi$, let $E\xi$ and $S\xi$ be the total spaces of the associated disc and sphere bundles, respectively. We construct a commutative diagram of pairs

$$
\begin{array}{ccc}
(E\tau, S\tau) & \overset{\phi}{\to} & (\mathbf{R}S^k, SS^k) \\
\downarrow\pi & & \downarrow\pi \\
(E\nu, S\nu) & \overset{\Upsilon}{\to} & (\mathbf{R}^*S^k, PS^k)
\end{array}
$$

as follows. $E\tau = \{(u, v) \in S^k \times B^{k+1} | u \perp v\}$, where $B^{k+1} \subset R^{k+1}$ is the unit ball, while $SS^k = S\tau = \{(u, v) \in S^k \times S^k | u \perp v\}$. For any $(u, v) \in E\tau$, let

$$\phi(u, v) = \begin{cases} (\alpha_v v + u, \alpha_v v - u) \in \mathbf{R}S^k & \text{if } (u, v) \notin S\tau, \\ (v, u) \in SS^k & \text{if } (u, v) \in S\tau, \end{cases}$$

where $\alpha_v = (1 - \|v\|^2)^{1/2}$, and let $\Upsilon$ be the unique map which makes the diagram commutative. Trivially, $\phi$ and $\Upsilon$ are both homeomorphisms of pairs. Thus $\mathbf{R}^* S^k / PS^k \cong M(v)$, the Thom complex of $v$, whence $H^i(\mathbf{R}^* S^k, PS^k; Z_2) \cong Z_2$ if $k \leqslant i \leqslant 2k$, 0 otherwise; and $Sq^j: H^i(\mathbf{R}^* S^k, PS^k; Z_2) \to H^{i+j}(\mathbf{R}^* S^k, PS^k; Z_2)$ is nonzero if and only if $k \leqslant i \leqslant i + j \leqslant 2k$ and $\binom{i-k-1}{j} = 1 \bmod 2$ (where we take $\binom{-1}{j} = 1 \bmod 2$ for all $j$). $\mathbf{R}S^k / SS^k$ may then be considered to have two cells, $e_1^i$ and $e_2^i$, in each dimension $k \leqslant i \leqslant 2k$, and none in other dimensions, such that:

(i) $Te_1^i = e_2^i$ for all $i$, where $T: \mathbf{R}S^k \to \mathbf{R}S^k$ is the map which exchanges coordinates;

(ii) For all $k \leqslant i < 2k$ and $j = 1$ or 2,

$$\delta e_j^i = \begin{cases} e_1^{i+1} + e_2^{i+1} & \text{if } i - k \text{ is even,} \\ (-1)^{j+1}(e_1^{i+1} - e_2^{i+1}) & \text{if } i - k \text{ is odd.} \end{cases}$$

Then $\mathbf{R}^* S^k / PS^k$ has one cell in each dimension $k \leqslant i \leqslant 2k$, namely $e^i = \pi e_1^i = \pi e_2^i$, and no cells in other dimensions; and $\delta e^i = (1 + (-1)^{i-k})e^{i+1}$.

Now let $G$ be any Abelian group and $e: G \to G$ an automorphism of order 2. Recall that $\pi_1(\mathbf{R}^* S^k) \cong T_2 = \{1, m\}$, a multiplicative group of order 2. Let $\mu_e: T_2 \times G \to G$ be the action where $\mu_e(m, x) = ex$ for all $x \in G$, and let $G^e = S(G, \mu_e)$, a sheaf over $\mathbf{R}^* S^k$. $H^*(\mathbf{R}^* S^k, PS^k; G^e)$ may be considered to be the equivariant cohomology of the pair $(\mathbf{R}S^k, SS^k)$ with coefficients in $G$ under the action $\mu_e$, specifically, the homology of the chain complex

$$G \xrightarrow{d^k} G \xrightarrow{d^{k+1}} G \to \cdots \to G \xrightarrow{d^{2k-1}} G$$

where, for $k \leqslant i < 2k$, $d^i x = x + (-1)^{i-k} ex$ for all $x \in G$. Letting $G_{n-1} = Z\pi_1$ and $ea = (-1)^a (-1)^n a^{-1}$ for all $a \in \pi_1$, and letting $G_n = Z\pi_1 \otimes (\pi_2 \oplus H_2)$ and $e(a \otimes (x + \lambda\eta)) = ((-1)^a(-1)^{n+1}a^{-1} \otimes x^a + a^{-1} \otimes (\lambda + w_2 x)\eta)$ for all $a \in \pi_1$, $x \in \pi_2$, $\lambda \in Z_2$, we have by 5.3.3 that $G_s^e = F^{-1}\pi_s \theta_M$ for $s = n - 1$ or $n$, and we are done with the proof of (I).

PROOF (II). For convenience of notation, we agree to let $S_s = F^{-1}\pi_s \zeta_M$ and $Q_s = F^{-1}\pi_s \theta_M$ for $s = n - 1$ or $n$. Let $L_s \subset S_s$ be the unique maximal subsheaf with a local product structure, specifically, $L_s = J_s^e$, where $J_{n-1} = Z \subset G_{n-1}$ and $J_n = Z \otimes H_2 \subset G_n$. Note that $G_s = J_s \oplus R_s$, for suitably chosen $R_s \subset G_s$, and $T_2$ acts independently on each direct summand: it follows that $S_s = L_s \oplus R_s^e$, a direct summation of sheaves. We also have that $L_s|PS^k = S_s|PS^k$ and $Q_s|\mathbf{R}^* S^k = S_s|\mathbf{R}^* S^k$, and:

$$L_n \cong Z_2;$$

$$L_{n-1} \cong \begin{cases} Z & \text{if } n \text{ is even,} \\ Z^T & \text{if } n \text{ is odd.} \end{cases}$$

Thus $i^*: H^r(\mathbf{R}^* S^k, PS^k; L_s) \to H^r(\mathbf{R}^* S^k, PS^k; S_s)$ is mono, its image

exactly what we wish $\mathrm{Ker}\, j^*$ to be. We have a commutative diagram with exact rows (since $H^i(\mathbf{R}^*S^k; L_s) = 0$ for all $i > k$, because $L_s$ has a local product structure and $\mathbf{R}^*S^k$ is of the homotopy type of $P^k$):

$$H^{r-1}(PS^k; L_s) \longrightarrow H^r(\mathbf{R}^*S^k, PS^k; L_s) \longrightarrow 0 \longrightarrow H^r(PS^k; L_s) \longrightarrow H^{r+1}(\mathbf{R}^*S^k, PS^k; L_s) \longrightarrow 0$$

$$\cong \bigg\downarrow \qquad 1-1 \bigg\downarrow i^* \qquad \qquad \qquad \bigg\downarrow \qquad \cong \bigg\downarrow \qquad 1-1 \bigg\downarrow i^*$$

$$H^{r-1}(PS^k; S_s) \xrightarrow{\ \delta\ } H^r(\mathbf{R}^*S^k, PS^k; S_s) \xrightarrow{\ j^*\ } H^r(\mathbf{R}^*S^k; S_s) \xrightarrow{\ i^*\ } H^r(PS^k; S_s) \xrightarrow{\ \delta\ } H^{r+1}(\mathbf{R}^*S^k, PS^k; S_s)$$

By a simple diagram-chasing argument, we are done.

*Notation.* If $y \in \mathrm{Ker}\, \delta^r$, we shall let $[y] \in \mathrm{Ker}\, \delta^r/\mathrm{Im}\, \delta^{r-1}$ be the class represented by $y$.

In the following lemmas, we shall assume that $n = 2k, k \geqslant 3$.

We examine a particular case. For any integer $s$, let $N_s^n$ be the total space of a real $(n-2)$-plane bundle $\xi_x^{n-2}$ over $P^2$ such that $\xi_s^{n-2} \oplus 3h$ is stably equivalent to $sh$, the $s$-fold Whitney sum of the canonical line bundle, $h$. Clearly, $N_s^n = N_{s+4}^n$, since $h$ has order 4 in $K$-theory. If $\tau_s$ is the tangent bundle of $N_s^n = N$, $\tau_s$ is stably equivalent to $sh$, hence $w_1 N = \binom{s}{1}m$ and $w_2 N = \binom{s}{2}m^2$, where $m \in H^1(N; Z_2)$ is the generator of that group.

Let $a \in \pi_1 N \cong T_2$ be the generator. Then $\tilde{a} \in \pi_2 N \cong Z$ is a generator, and $\tilde{a}^a = -\tilde{a}$. Let $f \colon S^k \to N$ be an embedding. Recall that $\Phi$ is the fiber of $\theta_N$. We have by 5.3.2 that

$$F^{-1}\pi_{n-1}\theta_N \cong \begin{cases} Z + Z & \text{if } s \text{ is even,} \\ Z + Z^T & \text{if } s \text{ is odd,} \end{cases}$$

where the first generator is represented by $1 \in Z\pi_1 = \pi_{n-1}\Phi$, the second by $a \in \pi_{n-1}\Phi$; and also that

$$F^{-1}\pi_n\theta_N \cong \begin{cases} Z^T + Z_2 + Z + Z_2 & \text{if } s \text{ is even,} \\ Z^T + Z_2 + Z^T + Z_2 & \text{if } s \text{ is odd,} \end{cases}$$

where the generators are represented, respectively, by $1 \otimes a$, $1 \otimes \eta$, $a \otimes \tilde{a}$, $a \otimes \eta \in Z\pi_1 \otimes (\pi_2 \oplus H_2) = \pi_n\Phi$.

Let $B = \mathbf{R}^*S^k$. Since $f$ is an embedding, $F$ has a lifting to $Y_N$, thus $F^{-1}\theta_N$ has a section, i.e. $F^{-1}\theta_N$ is in the category $\mathcal{X}_B^+$ of $B$-sectioned fibrations. The following lemma is phrased in the notation of [10].

LEMMA 6.1.2. *If $N = N_s^n$, then the first two stages of the Postnikov tower for $F^{-1}\theta_N$ are given by* ($B = \mathbf{R}^*S^k$):

$$k_B(Z, n, m) \times k_B(Z_2, n) \times k_B\!\left(Z, n, \binom{s}{1}m\right) \times k_B(Z_2, n) \xrightarrow{\ \lambda\ } e_1 \xleftarrow{\ P_1\ } F^{-1}\theta_N$$

$$\bigg\downarrow$$

$$k_B(Z, n-1) \times k_B\!\left(Z, n-1, \binom{s}{1}m\right) = e_0 \xrightarrow{\ \alpha\ }$$

$$k_B(Z, n+1, m) \times k_B(Z_2, n+1) \times k_B\!\left(Z, n+1, \binom{\overline{s}}{1}m\right) \times k_B(Z_2, n+1)$$

where $P_1$ induces isomorphism in homotopy through dimension $n$, and where

$$\alpha^*: \iota_{n+1} \otimes 1 \otimes 1 \otimes \mapsto 0,$$

$$1 \otimes \iota_{n+1} \otimes 1 \otimes 1 \mapsto (\mathrm{Sq}^2 + m^2)\iota_{n-1} \otimes 1,$$

$$1 \otimes 1 \otimes \iota_{n+1} \otimes 1 \mapsto 1 \otimes (\delta m)\iota_{n-1},$$

$$1 \otimes 1 \otimes 1 \otimes \iota_{n+1} \mapsto 1 \otimes \left(\mathrm{Sq}^2 + \tbinom{n-s}{2}m^2\right)\iota_{n-1},$$

where $\delta$ is the Bokstein homomorphism associated with the coefficient sequence $Z \to^2 Z \to Z_2$.

We postpone the proof.

LEMMA 6.1.3. *Let* $a \in \pi_1$, $a^2 = 1$, *and choose (jointly)* $a \in \pi_2$, $0 < s = k_a < 4$ (*cf.* 5.2). *Then in sequence* (6.1-1):

(I) *If* $(-1)^a(-1)^k = -1$, $d_2[a] = a \otimes \tilde{a} + ((\tbinom{k-3}{2}) + (\tbinom{n-s}{2}))a \otimes \eta$;

(II) *If* $(-1)^a(-1)^k = 1$, $\Phi_2[a] \in \mathrm{Coker}\, d_2$ *is represented by* $[a \otimes \tilde{a} + ((\tbinom{k-3}{2}) + (\tbinom{n-s}{2}) + (\tbinom{s}{1}))a \otimes \eta]$.

PROOF. We remind the reader that $(-1)^a = (-1)^s$, thus $\binom{s}{1} = \binom{k}{1}$ mod 2 if $(-1)^a(-1)^k = 1$.

Without loss of generality, $f$ is inessential, hence we may assume $f: S^k \to N_s^n \subset M$. Since all constructions involved are natural with respect to inclusions of manifolds of the same dimension, we may assume that $M = N_n^s$. We have a commutative diagram with exact rows, where the first, second and fourth vertical arrows represent onto maps, by 6.1.1:

$$
\begin{array}{ccccccccc}
H_{n-1}^{n-2} & \xrightarrow{d_2} & H_n^n & \xrightarrow{\lambda} & [\mathbf{R}^*S^k, PS^k; \theta_M]_F & \xrightarrow{\rho} & H_{n-1}^{n-1} & \to 0 \\
\text{onto}\downarrow j^* & & \text{onto}\downarrow j^* & & \downarrow j^* & & \text{onto}\downarrow j^* & & \\
E_2^{-1,n-1} & \xrightarrow{d_2} & E_2^{0,n} & \xrightarrow{\lambda} & [S^k \subset M]_f & \xrightarrow{\rho} & E^{0,n-1} & \to 0
\end{array}
$$

It is sufficient to compute $d_2$ and $\Phi_2$ for the top row. Recall that $\cup m^2: H^{n-2}(\mathbf{R}^*S^k, PS^k; Z_2) \to H^n(\mathbf{R}^*S^k, PS^k; Z_2)$, while $\mathrm{Sq}^2$, on the same group, is nonzero if and only if $\binom{k-3}{2} = 1$ mod 2. Part (I) now follows immediately from 6.1.1 and 6.1.2, while (II) follows from 6.1.1, 6.1.2, and the results of [10].

6.2. *Proof of* 6.1.2. Let $p: \tilde{N} \to N$ be the universal covering of $N$; $\tilde{N} = S^2 \times R^{n-2}$. Since $f$ may be assumed to be inessential, we may choose an embedding $\tilde{f}: S^k \to N$ such that $p\tilde{f} = f$. Note that $R^\infty \cong R^\infty \times R^\infty$. Thus we may choose an embedding $P$ such that the following diagram commutes, where $p_1$ is projection:

$$
\begin{array}{ccc}
\tilde{N} \times R^\infty & \xrightarrow{P} & N \times R^\infty \\
\downarrow p_1 & & \downarrow p_1 \\
\tilde{N} & \xrightarrow{p} & N
\end{array}
$$

Let $\nabla N = (p^2)^{-1} \nabla N - \Delta \tilde{N} \subset R\tilde{N}$, and let $\nabla^* N \subset R^* \tilde{N}$ be the image of $\nabla N$. Let $\tilde{R}N = R\tilde{N} - \nabla N$ and $\tilde{R}^* N = R^* N - \nabla^* N$. Consider the diagram (where $p^*[x, y] = [px, py]$ for all $[x, y] \in \tilde{R}^* N$):

$$
\begin{array}{ccc}
\tilde{R}^* N & \overset{p}{\rightarrow} & R^* N \\
\cap & & \\
R^* \tilde{N} & & \cap \\
\cap & & \\
R^*(\tilde{N} \times R^\infty) & \overset{R^* P}{\rightarrow} & R^*(N \times R^\infty)
\end{array}
$$

We have that $\Phi = $ fiber of $\tilde{R}^* N \subset R^*(\tilde{N} \times R^\infty) = $ fiber of $R^* N \subset R^*(N \times R^\infty)$.

LEMMA 6.2.1 (HAEFLIGER [4]). (I) *As an algebra over* $Z_2$, $H^* = H^*(R^*(\tilde{N} \times R^\infty); Z_2) = H^*(\Gamma S^2; Z_2)$ *is generated only by the elements:* $m \in H^1$, $\Gamma \sigma \in H^2$, *and* $\Sigma \sigma \in H^4$, *subject only to the relations:* $m\Gamma\sigma = (\Gamma\sigma)^2 = \Gamma\sigma\Sigma\sigma = (\Sigma\sigma)^2 = 0$.

(II) $(R^* i)$: $H^* \rightarrow H^*(R^* N; Z_2)$ *is surjective, and its kernel is generated by* $m^n$ *and* $m^{n-2}\Sigma\sigma$.

LEMMA 6.2.2. *The map* $i^*$: $H^*(R^* \tilde{N}; Z_2) \rightarrow H^*(\tilde{R}^* N; Z_2)$ *is injective, and its cokernel is generated over* $Z_2$ *by* $\omega \in H^{n-1}(\tilde{R}^* N; Z_2)$, $m\omega$, *and* $m^2\mu$. *Furthermore,* $(\Gamma\sigma)\omega = m^2\omega$, *and* $Sq^i\omega = \binom{n-s}{i}m^i\omega$ *for all* $i$.

PROOF. We construct a map $\beta$: $R^* \tilde{N} \rightarrow \tilde{R}^* N$ such that $i\beta$ is homotopic to the identity on $R^* \tilde{N}$. Recall that $N = N_s^n$, the total space of $\xi_s^{n-2} = \xi$ over $P^2$. Choose a nonzero section $\chi^*$ of $\xi \otimes h$, and let $\chi$: $S^2 \rightarrow S^2 \times (R^{n-2} - \{0\})\tilde{N}$ be the corresponding section of $\pi^{-1}(\xi \otimes h)$, the trivial $n$-bundle over $S^2$. Let $p_1$: $\tilde{N} \rightarrow S^2$, $p_2$: $\tilde{N} \rightarrow R^{n-2}$ be the projections. For all $0 \leq t \leq 1$, let $\beta_t$: $R^* \tilde{N} \rightarrow \tilde{R}^* N$ be defined as follows. If $[x, y] \in R^* \tilde{N}$, let $\rho = \rho(x, y) = \|p_1 x - p_1 y\|$, and let:

$$
\beta_t[x, y] = \begin{cases} [x, y] & \text{if } 1 \leq \rho \leq 2, \\ \big[(p_1 x, (1 + (\rho - 1)t)p_2 x + (1 - \rho)tp_2(\chi(p_1 x))), \\ \quad p_1 y, (1 + (\rho - 1)tp_2 y + (1 - \rho)tp_2(\chi(p_1 y)))\big] \\ \hfill \text{if } 0 \leq \rho \leq 1. \end{cases}
$$

Let $\beta = \beta_1$, which clearly has the desired property. Now the cofiber of the inclusion $\tilde{R}^* N \subset R^* \tilde{N}$ is the Thom complex of $\tau \otimes h$ over $N$, which we denote by $TC$, where $\tau$ is the tangent bundle of $N$. We have a commutative diagram with split exact columns, where each row is a long exact Thom-Gysin sequence, and all coefficients are $Z_2$:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \rightarrow & H(R^*\widetilde{N}) & \xrightarrow{\pi^*} & H(R\widetilde{N}) & \xrightarrow{\theta} & H^*(R^*\widetilde{N}) & \xrightarrow{\cup m} & H(R^*\widetilde{N}) & \rightarrow \cdots \\
& \beta^* \Big\Updownarrow i^* & & \beta^* \Big\Updownarrow i^* & & \beta^* \Big\Updownarrow i^* & & \beta^* \Big\Updownarrow i^* & \\
\cdots \rightarrow & H^*(\widetilde{R}^*N) & \xrightarrow{\pi^*} & H^*(\widetilde{R}N) & \xrightarrow{\theta} & H^*(\widetilde{R}^*N) & \xrightarrow{\cup m} & H^*(\widetilde{R}^*N) & \rightarrow \cdots \\
& \delta \Big\downarrow & & \delta \Big\downarrow & & \delta \Big\downarrow & & \delta \Big\downarrow & \\
\cdots \rightarrow & H^*(TC) & \xrightarrow{\pi^*} & H^*(\widetilde{N}/S^2) & \xrightarrow{\theta} & H^*(TC) & \xrightarrow{\cup m} & H^*(TC) & \rightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 & & 0 &
\end{array}
$$

We now define $\omega \in H^{n-1}(\widetilde{R}^*N; Z_2)$ to be the unique element such that $\beta^*\omega = 0$ and $\delta\omega = U$, the Thom class of $\tau \otimes h$. Now $H^*(TC)$ has only generators $U$, $mU$, and $m^2U$ over $Z_2$, and $\mathrm{Sq}^iU = \binom{n-s}{i}m^iU$ for all $i$, although $m^3U = 0$. By exactness of the third row, $\theta(\sigma \cup \pi^*U) = \theta((\sigma \otimes 1) \cup \pi^*U)$ must be $m^2U$. Also, $\theta(\sigma \otimes 1) = \Gamma\sigma$, thus $(\Gamma\sigma)\omega = \theta((\sigma \otimes 1) \cup \pi^*\omega) = m^2\omega$. 6.2.2 then follows trivially.

Finally, Lemma 6.1.2 follows from routine Serre spectral sequence computation of the relative Postnikov tower of the inclusion $\widetilde{R}^*N \subset R^*(\widetilde{N} \times R^\infty)$.

### 7. Affine groups.

DEFINITION 7.1. An *affine group* is a set $A$ together with an operation $\cdot_a$ for each $a \in A$, such that $(A, \cdot_a)$ is a group with identity $a$, and such that, for any $x, y, a, b \in A$, $x \cdot_a y = x \cdot_b (a)_b^{-1} \cdot_b y$, where $(a)_b^{-1}$ is the inverse of $a$ under $\cdot_b$. In addition, we say that $A$ is *Abelian* if $\cdot_a$ is commutative for all $a \in A$. (The operation shall then usually be denoted $+_a$.)

We remark that Becker's definition of affine group [1] is that of Abelian affine group.

An alternative definition, which is more natural to the constructions of this paper, is:

DEFINITION 7.2. An *affine group* is a set together with a ternary operation $\tau$ on $A$ such that (writing $\tau(x, y, z) = xy^{-1}z$)

(i) $vw^{-1}(xy^{-1}z) = (vw^{-1}x)y^{-1}z$ (associative law),

(ii) $xy^{-1}y = yy^{-1}x = x$ (cancellation law).

In addition, we say that $A$ is *Abelian* if

(iii) $xy^{-1}z = zy^{-1}x$ (commutative law).

We leave the proof of the equivalence of 7.1 and 7.2 to the reader as an exercise (Hint: $x \cdot_a y = xa^{-1}y$). Note that the empty set is an affine group.

Letting $A$ be any affine group, we define two equivalence relations on $A \times A$, $R$ and $L$. For any $x, y, z, w \in A$, we say that $(x, y)R(z, w)$ if $y = xz^{-1}w$, and $(x, y)L(z, w)$ if $x = zw^{-1}y$. (Note that $R = L$ if $A$ is commutative.) Let $A^R = A \times A/R$, and $A^L = A \times A/L$. If $x, y \in A$, let $x^{-1}y \in A^R$ and $xy^{-1} \in A^L$ denote the elements represented by the ordered pair $(x, y)$. Both $A^R$ and $A^L$ are groups, under the operations $(x^{-1}y)(z^{-1}w) = x^{-1}(yz^{-1}w)$, and $(xy^{-1})(zw^{-1}) = (xy^{-1}z)w^{-1}$, respectively. We call $A^R$ and $A^L$ the right and left action groups of $A$, since $A^R$ acts on $A$ on the right, in the obvious manner suggested by the notation, and $A^L$ acts on $A$ on the left. We remark that $A^R$ and $A^L$ are isomorphic, canonically if $A$ is Abelian, in which case we write $A^* = A^R = A^L$, and $x - y$ instead of $xy^{-1}$ or $y^{-1}x$.

We remark that two affine groups are isomorphic if and only if their action groups are isomorphic. If $G$ is any group, $G$ may be taken to be an affine group by letting $xy^{-1}z$ have the usual meaning ($x - y + z$ if $G$ Abelian), in which case $G \cong G^L$ by letting $x$ correspond to $x1^{-1}$.

If we say that an affine group $A$ is isomorphic to a group $G$, we shall mean that $A$ is isomorphic to $G$ considered as an affine group; equivalently, $A^R \cong G$ or $A^L \cong G$.

**8. Correction.** Recall diagram (2.6-1) in this paper, which is basically diagram (3.2-1) of [7]. We define a pair fibration $'\zeta_M: ('Y_M, 'Z_M) \to (^\infty RM, ^\infty PM)$ as follows: Let $'Y_M$ be the space consisting of all paths $\sigma: I \to {}^\infty R^*M$ such that $\sigma(1) \in RM$, and either $\sigma(0) \in {}^\infty PM$ or $\sigma(1) \notin PM$. Let $'Z_M$ be the space consisting of all paths $\sigma: I \to {}^\infty PM$ such that the composition $I \xrightarrow{\sigma} {}^\infty PM \to M$ is constant. Let $'\zeta_M: ('Y_M, 'Z_M) \to (^\infty R^*M, ^\infty PM)$ be evaluation at 0, a pair fibration, and let $\iota: '\zeta_M \subset \zeta_M$ be the obvious inclusion.

LEMMA 8.1. $\iota_\#: [(\mathbf{R}^*V, PV); '\zeta_M]_F \cong [(\mathbf{R}^*V, PV); \zeta_M]_F$.

PROOF. Let $''\zeta_M: ('Y_M, Z_M) \to (^\infty R^*M, ^\infty PM)$. Now the inclusion $i: ('Y_M, Z_M) \subset (Y_M, Z_M)$ is a fiber homotopy equivalence of pairs. Let $U \subset {}^\infty R^*M$ be a regular collar of $^\infty PM$, such that $U \cap RM$ is a regular collar of $PM$. Let $r_t: {}^\infty R^*M \to R^*M$, $0 \le t \le 1$, with $r_0 = $ identity, be a strong deformation retraction of $^\infty R^*M$ onto the complement of $U$ such that $r_t(R^*M) \subset R^*M$ for all $t$. Let $\rho: {}^\infty R^*M \to I$ be continuous such that $\rho^{-1}\{0\} = {}^\infty PM$, and let $(j\sigma)(t) = r_s(t)$ for all $\sigma \in Y_M$, $t \in I$, where $s = t\rho(\sigma(0))$. Clearly $j$ is the identity on $^\infty PM$. The fact that $j$ is a pair fiberwise homotopy inverse of $i$ is trivial to verify. Thus $i_\#: [(\mathbf{R}^*V, PV); ''\zeta_M]_F \cong [(\mathbf{R}^*V, PV); \zeta_M]_F$. Now $\pi_k('\zeta_M) = \pi_k(''\zeta_M)$ for all $k$, since $'Z_M \to {}^\infty PM$ and $Z_M \to {}^\infty PM$ both have fibers of the homotopy type of $S^{n-1}$. By 2.4.1, we are done.

The error in the proof of Theorem 3.3.1 of [7] is the fact that if a lifting $\Phi$

of $F$ to $\zeta_M$ is given, the map $G[\Phi]$ in the  diagram on p. 362 may not exist. If, on the other hand, $\Phi$ is a lifting of $F$ to $'\zeta_M$ (which we may assume, by Lemma 8.1), existence of $G[\Phi]$ is assured. The remainder of the proof is valid.

Theorem 3.3.2 of [7] can be corrected in a similar manner.

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